GLOBAL REGULARITY OF WAVE MAPS FROM $\mathbf{R^{2+1}}$ TO $\mathbf{H^2}$. SMALL ENERGY

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Abstract. We demonstrate that Wave Maps with smooth initial data and small energy from $\mathbf{R^{2+1}}$ to the Lobatchevsky plane stay smooth globally in time. Our method is similar to the one employed in [18]. However, the multilinear estimates required are considerably more involved and present novel technical challenges. In particular, we shall have to work with a modification of the functional analytic framework used in [30], [33], [18].

1. Formulation of the problem and overview.

Let $(M,\, g)$ be a Riemannian manifold equipped with metric $g = (g_{ij})$. Also, let $\mathbf{R}^{n+1}$, $n \geq 1$, be the standard Minkowski space equipped with metric

$$(\delta_{ij}) = \mathrm{diag}(-1, 1, \dots, 1).$$

A **classical Wave Map** u from $\mathbf{R}^{n+1}$ to $(M,\, g)$ is a smooth map which is critical with respect to the functional

$$u \to \int_{\mathbf{R}^{n+1}} < \partial_\alpha u, \partial^\alpha u >_g \, d\sigma.$$

The following notational conventions are used: $d\sigma$ denotes the volume measure associated with (δ_{ij}), $\partial_\alpha u = u_*(\partial_\alpha) \in TM$, $\alpha = 0, 1, \dots, n$, and Einstein's summation convention is in force.[1] Moreover, $\partial_\alpha = \delta_{\alpha\beta}\partial^\beta$. In local coordinates, u is seen to satisfy the following conditions:

$$\Box u^i + \Gamma^i_{jk}\partial_\alpha u^j \partial^\alpha u^k = 0 \tag{1}$$

where $u = (u^i)$, and Γ^i_{jk} are the Riemann-Christoffel symbols associated with the metric g and the local coordinate system. We are interested in the

Cauchy problem associated with (1)**:**
Given smooth initial data $u[0] := (u(0), \partial_t u(0)) : 0 \times \mathbf{R}^2 \to M \times TM$ at time $t = 0$,

[1] We shall also use the convention $\partial_0 = \partial_t$

is there a Wave Map $u(t, x)$ extending these globally in time?

To start with, we observe that the problem is supercritical with respect to the conserved energy provided $n > 2$.[2] Thus, one expects development of singularities for 'large initial data'. Blow-up examples are given for instance in [25], p. 102. Still, in sync with the general philosophy developed by Klainerman e. g. in [7], one expects existence of classical Wave Maps provided the initial data are smooth and small in the critical Sobolev space $\dot{H}^{\frac{n}{2}}$. Moreover, for the energy critical case $n = 2$, one hopes for existence of classical Wave Maps for arbitrary smooth data, provided the target (M, g) is 'sufficiently nice'. Of particular importance is the following conjecture of Klainerman, in light of its close connection to Einstein's equations under $U(1)$-symmetry[3] :

Conjecture (Klainerman): Let $(\mathbf{H}^2, dg)$ be the standard hyperbolic plane. Then classical Wave Maps originating on $\mathbf{R}^{2+1}$ exist for arbitrary smooth initial data.

Furthermore, numerical evidence elaborated in [2] suggests development of singularities for Wave Maps from $\mathbf{R}^{2+1}$ to S^2, provided the (smooth) data are sufficiently large, even under certain symmetry assumptions (equivariance) on the Wave Map. In this paper, we shall establish a partial result towards the conjecture stated above, namely the following **small data result:**

Theorem 1.1. *Let $\mathbf{H}^2$ be the standard hyperbolic plane, consisting of all pairs of real numbers $\{(\mathbf{x}, \mathbf{y})|\mathbf{y} > 0\}$ equipped with the metric $dg = \frac{d\mathbf{x}^2+d\mathbf{y}^2}{\mathbf{y}^2}$. Then, given smooth initial data $(\mathbf{x}, \mathbf{y})[0] : 0 \times \mathbf{R}^2 \to \mathbf{H}^2$ which are sufficiently small in the sense that*

$$\int_{0\times\mathbf{R}^2} \sum_{\alpha=0}^{2} ([\frac{\partial_\alpha \mathbf{x}}{\mathbf{y}}]^2 + [\frac{\partial_\alpha \mathbf{y}}{\mathbf{y}}]^2) dx < \epsilon$$

for suitably small $\epsilon > 0$, there exists a classical Wave Map from $\mathbf{R}^{2+1}$ to $\mathbf{H}^2$ extending these globally in time.

This result is to be seen as a further step in a long sequence of developments, whose high points are the following achievements:

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[2]To define the energy, for example isometrically embed (M, g) into an ambient Euclidean space using Nash's embedding theorem, and put $||u||^2_{H^1} := \sum_{\alpha=0}^{n} \int_{t=const} |\partial_\alpha u(t, .)|^2 d\sigma$, where $|,|$ denotes Euclidean length. This is easily seen to be a well-defined quantity for classical Wave Maps.

[3]Einstein's equations in vacuo under $U(1)$-symmetry attain the form of a Wave Map originating on a Lorentzian $2 + 1$-manifold M to $\mathbf{H}^2$, coupled with an elliptic system driving the metric on M. Our form of the Wave Maps problem is a highly simplified version.

(1) The **subcritical case** for $n \geq 2$: strong local well-posedness of (1) in H^s, $s > \frac{n}{2}$ by Klainerman-Machedon [9] and Klainerman-Selberg [13] .[4]

(2) The **critical Besov case** for $n \geq 2$: strong global well-posedness of (1) for initial data small in the critical Besov space $\dot{B}^{\frac{n}{2},1}$ by Tataru [33].

(3) **Global regularity** for Wave Maps from $\mathbf{R}^{n+1}$, $n \geq 2$ to S^k, $k \geq 1$, provided the initial data are smooth and small in the critical Sobolev space $\dot{H}^{\frac{n}{2}}$ by Tao [29], [30].

(4) Extension of preceding result to the case $n \geq 5$ and **more general targets** (boundedly parallelizable, compact) by Klainerman-Rodnianski [12].

(5) **Massive simplification** and extension of the previous case to $n \geq 4$ by Shatah-Struwe [24].

(6) **Ill-posedness** of the Cauchy problem in H^s, $s < \frac{n}{2}$ by d'Ancona-Georgiev [1].

Further developments include an alternative proof of (5) (in more restrictive formulation) by Nahmod-Stefanov-Uhlenbeck [20][5], as well as extension of (5) to the case $n \geq 3$ by the author in [17], [18], [19]. Also, a recent preprint by D.Tataru [34] (which appeared when the research for this paper had been concluded) promises to solve the small-data case for $n \geq 2$ and targets which can be uniformly isometrically imbedded into some Euclidean space. This condition appears to fail for the hyperbolic plane, as it would require at most polynomial growth of disc areas with respect to their radius. Adding some comments on the above-listed developments, we observe that in (1) the crucial **framework of $X^{s,b}$ spaces in the context of the wave equation** was developed, and the **null-structure** of the schematic type $Q_0(u, v) = \partial_\nu u \partial^\nu v$ of the nonlinearity was exploited. This didn't suffice, however, to settle the critical case, and (2) involved further sophisticated Banach spaces employing **decompositions into travelling waves.** These were the main harmonic analysis ingredients that went into (3) (provided that $n \leq 4$; in the case $n > 4$, Strichartz type spaces sufficed); the important additional features of the work of Tao, however, were the use of an inherent **Gauge freedom** in the problem, as well as **sophisticated trilinear null-form estimates.** Construction of a suitable Gauge, in turn, depended upon taking advantage of a **hidden skew-symmetry** in the equations, which attain the following form when the target is S^k:

$$\Box u = -u \partial_\alpha u^t \partial^\alpha u,\, u \in S^k \subset \mathbf{R}^{k+1}.$$

The 'skew-symmetry' is evidenced by the equality[6]

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[4]The problem for $n = 1$ is globally strongly well-posed in H^1, [5]. However, it is not well-posed in the critical $H^{\frac{1}{2}}$ [28].

[5]This team also recently announced the $3 + 1$ case, provided the target is a symmetric space.

[6]The matrix $(u \partial_\alpha u^t - \partial_\alpha u u^t)$ occuring on the right-hand side is skew-symmetric.

$$u\partial_\alpha u^t \partial^\alpha u = (u\partial_\alpha u^t - \partial_\alpha u u^t)\partial^\alpha u.$$

The Gauge was used in order to eliminate those frequency interactions for which u has the smallest of all occuring frequencies. In (4), the 'extrinsic formulation' via embedding the target isometrically was abandoned, and replaced by an 'intrinsic approach'[7], parametrizing the Wave Map by means of variables $\{\phi_\alpha^i\}$, which express the **derivatives of the Wave Map** $\partial_\alpha u$, in terms of a global orthonormal frame $\{e_i\}$ for TM, provided the latter is a trivial bundle: $\partial_\alpha u = \phi_\alpha^i e_i$. One then obtains a first order system of equations of **divergence-curl** type for the ϕ_α^i, which in turn lead to Wave equations of the schematic form

$$\Box\phi = \phi\nabla\phi + \phi^3.$$

While the nonlinearity in the above is not amenable to estimation, Klainerman-Rodnianski exploited a further skew-symmetry (hinging on the orthogonality of the frame $\{e_i\}$), as well as the introduction of a **partial Coulomb Gauge** (which in turn takes advantage of the freedom in choosing the frame $\{e_i\}$), in order to modify the nonlinearity: more precisely, singling out the 'bad frequency interactions' in the nonlinearity[8], they choose a Gauge which allows elimination of just these. The remaining frequency interactions in the nonlinearity can then be estimated by means of Strichartz type norms, provided $n \geq 5$, without invocation of any null-structures (as in the 'high-dimensional' work [29] ($n \geq 5$) of Tao). In (5), Shatah-Struwe observed that using a **global Coulomb Gauge**, one could immediately modify the form of the equations schematically to the following:

$$\Box\phi = \nabla^{-1}(\phi^2)\nabla_{x,t}\phi$$

The nonlinearity here can be estimated for $n \geq 4$ by means of somewhat non-standard Strichartz type estimates (involving Lorentz spaces), without further use of microlocalization. A particularly transparent proof results. The previous work of the author [17], [18], [19], combined this approach (global Coulomb Gauge) together with the functional analytic framework of Tataru and multilinear null-form estimates of the type considered by Tao in [30], to settle the case $n = 3$. The null-structure in turn was rendered visible by using the device of **dynamic separation**[9], exploiting a Hodge-type decomposition of the variables ϕ_α^i. While the estimates are messy, they are not as tough as the ones in [30] for the case $n = 2$, as the linear theory is better in this somewhat higher dimensional setting.

In the present paper, we extend these investigations to the case $n = 2$ and target $\mathbf{H}^2$. While parts of the method appear to carry over to more general targets

[7]This was originally introduced in [3].

[8]Corresponding to the case when ϕ is at much lower frequency than $\nabla\phi$.

[9]This terminology was suggested by S. Klainerman; it was already used in a different (bilinear) context in [11].

(the cancellations we require stem from a quite general relation between Jacobi- and Christoffel symbols[10]), we prefer to stick to the present case on account of its transparency. One expects to obtain the same result for targets of bounded geometry, i. e. all covariant derivatives of the curvature tensor with respect to arbitrary slowly varying unit vector fields need to be globally bounded[11]. This would in particular encompass the hyperbolic plane as well as the class of targets considered in Tataru's recent preprint, and is possibly in some sense optimal. The nontrivial additional difficulty has to do with the fact that in general, the intrinsic approach no longer leads to an autonomous system, as is the case for the hyperbolic plane.
Our approach in this paper is to use the differentiated formulation of the Wave Maps problem. As mentioned before, Wave Maps to the hyperbolic plane have the pleasant property that going to the differentiated problem allows one to deduce an autonomous system which no longer involves the actual Wave Map u. In particular, one can avoid Moser type estimates. Also, the construction of a global Coulomb Gauge a la Shatah-Struwe is simple and explicit, thanks to the fact that $SO(2)$ is abelian. This focuses the difficulty purely on the null-form estimates, which are qualitatively distinct from the ones in [30], since we lose one degree of smoothness for large frequencies. This is in marked contrast with Tataru's recent approach, which uses an embedded formulation of the problem, without going to the derivative:

$$\Box u^i = S^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k,$$

where S^i_{jk} is the 2nd fundamental form of the isometric embedding into some Euclidean space. Tataru demonstrates that on account of the fact that essentially the same cancellation occurs here as for the sphere ($S^i_{jk}(u)\partial_\alpha u^i = 0$), one can rely on the same Gauge construction and trilinear estimates as the ones in [30], provided the target is uniformly isometrically embeddable into some Euclidean space. The novelty of Tataru's approach is thus more analytic, and in particular relies heavily on Moser type estimates.
While the overall strategy in the present paper is quite similar to the one pursued in [18], [17], we have to take into account additional structures in the nonlinearity. These have to do with the elementary observation that 'Coulomb-Gauging' the ϕ^i_α not only improves the resulting Wave equations, **but also the underlying divergence-curl system**. This allows us to formulate the Wave Map system in the form (8). We shall then use the device of **dynamic separation** to decompose the nonlinearity into various null-forms and error terms, which are analyzed by methods similarly to [30], [18]. As already mentioned, the main difference with respect to [30] is that we work at the level of the **derivative** of the Wave Map. This loss of smoothness forces us to modify the spaces employed in [30], [18], using finer decompositions ('discs' instead of 'angular sectors') on the Fourier side, since high-high frequency interactions become harder to analyze[12]. The overall technical

[10] Letting $\{e_i\}$ be an orthonormal frame as in the preceding discussion, and letting $\nabla_{e_j} e_k = \Gamma^i_{jk} e_i$, $[e_j, e_k] = C^i_{jk} e_i$, the identity we need is $C^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$.

[11] More precisely, we need $|\nabla_{e_1}\nabla_{e_2}...\nabla_{e_a} R_{ijkl}| < C$ provided $|\nabla_{e_i} e_j| < C$ and $|e_r| = 1$. Possibly it suffices to require this condition for only finitely many derivatives.

[12] This was not necessary in $3+1$ dimensions, since the linear theory provides better estimates.

scheme underlying this paper is similar to the one introduced by Tao in [29], [30]: we bootstrap the energies of the frequency localized pieces $||P_k\phi^i_\alpha||_{L^2_x}$, $k\in\mathbf{Z}$ on every time slice $t=const$[13]. Gaining global control over the **distribution of energy amongst the frequencies** then allows us to control some subcritical norm $||\phi||_{H^\sigma}$, $\sigma>0$, which by means of the subcritical result (1) proves the Theorem 1.1.

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2. Wave Maps to $\mathbf{H}^2$

We identify $\mathbf{H}^2$ with the upper half-plane and standard metric as before. Choose the orthonormal frame $\{e_{1,2}\}=\{-\mathbf{y}\partial_{\mathbf{x}},\,-\mathbf{y}\partial_{\mathbf{y}}\}$. We obtain a representation of the derivatives of the Wave Map as follows:

$$\partial_\alpha u=\sum_{i=1,2}\phi^i_\alpha e_i.$$

Proceeding as in [12], [18], one deduces the following **divergence-curl system**, provided u is a Wave Map:

$$\partial_\beta\phi^1_\alpha-\partial_\alpha\phi^1_\beta=\phi^1_\alpha\phi^2_\beta-\phi^1_\beta\phi^2_\alpha \tag{2}$$

$$\partial_\beta\phi^2_\alpha-\partial_\alpha\phi^2_\beta=0 \tag{3}$$

$$\partial_\alpha\phi^{1\alpha}=-\phi^1_\alpha\phi^{2\alpha} \tag{4}$$

$$\partial_\alpha\phi^{2\alpha}=\phi^1_\alpha\phi^{1\alpha}. \tag{5}$$

We pass to complex notation and introduce the **Coulomb Gauge**:

$$\psi_\alpha=\psi^1_\alpha+i\psi^2_\alpha:=(\phi^1_\alpha+i\phi^2_\alpha)e^{-i\triangle^{-1}\sum_{j=1}^2\partial_j\phi^1_j},$$

where $\triangle^{-1}$ is given by convolution with the standard Green's function on $\mathbf{R}^2$. One

[13] P_k denote standard Littlewood-Paley multipliers

then easily verifies the **fundamental divergence-curl system**:

$$\partial_\alpha\psi_\beta - \partial_\beta\psi_\alpha = i\psi_\beta\triangle^{-1}\sum_{j=1,2}\partial_j(\psi_\alpha^1\psi_j^2 - \psi_\alpha^2\psi_j^1) - i\psi_\alpha\triangle^{-1}\sum_{j=1,2}\partial_j(\psi_\beta^1\psi_j^2 - \psi_\beta^2\psi_j^1) \tag{6}$$

$$\partial_\nu\psi^\nu = i\psi^\nu\triangle^{-1}\sum_{j=1,2}\partial_j(\psi_\nu^1\psi_j^2 - \psi_\nu^2\psi_j^1). \tag{7}$$

One deduces the following Wave Equations:

$$\begin{aligned}\Box\psi_\alpha =& i\partial^\beta[\psi_\alpha\triangle^{-1}\sum_{j=1}^2\partial_j[\psi_\beta^1\psi_j^2 - \psi_\beta^2\psi_j^1]] \\ &- i\partial^\beta[\psi_\beta\triangle^{-1}\sum_{j=1}^2\partial_j[\psi_\alpha^1\psi_j^2 - \psi_\alpha^2\psi_j^1]] \\ &+ i\partial_\alpha[\psi_\nu\triangle^{-1}\sum_{j=1}^2\partial_j[\psi^{1\nu}\psi_j^2 - \psi^{2\nu}\psi_j^1]].\end{aligned} \tag{8}$$

As in [18], the nonlinearity here does not display an obvious null-structure. In order to render it visible, we apply **dynamic decomposition** to the variables ψ_α, by writing

$$\psi_\nu = -R_\nu\sum_{k=1}^2 R_k\psi_k + \chi_\nu,$$

where $R_\alpha := (\sqrt{-\triangle_x})^{-1}\partial_\alpha$, $\alpha = 0,1,2$, is a Riesz type operator. Substituting the 'hyperbolic terms' $R_\nu\sum_{k=1}^2 R_k\psi_k$ into the right-hand side of (8) results in a trilinear expression featuring a null-structure. On the other hand, substituting at least one 'elliptic' term χ_ν yields quintilinear terms, upon noting that

$$\sum_{j=1,2}\partial_j\chi_j = 0, \tag{9}$$

$$\partial_i\chi_\nu - \partial_\nu\chi_i = \partial_i\psi_\nu - \partial_\nu\psi_i, \tag{10}$$

whence

$$\chi_\nu = i\sum_{i,j=1}^2\partial_i\triangle^{-1}(\psi_\nu\triangle^{-1}\partial_j(\psi_i^1\psi_j^2 - \psi_j^1\psi_i^2) - \psi_i\triangle^{-1}\partial_j(\psi_\nu^1\psi_j^2 - \psi_j^1\psi_\nu^2)). \tag{11}$$

Indeed, one obtains expressions of the following schematic type:

$$\nabla(\nabla^{-1}(\psi\nabla^{-1}(\psi^2))\nabla^{-1}(\psi^2)) \tag{12}$$

$$\nabla(\psi\nabla^{-1}(\nabla^{-1}(\psi\nabla^{-1}(\psi^2))\psi)) \tag{13}$$

The basic idea is that **the quintilinear terms should be easier to estimate than the trilinear ones, and should essentially be amenable to estimation by means of Strichartz type norms.** Unfortunately, this is strictly speaking only true for the first quintilinear expression, and we have been unable to find an elegant method for dealing with the second. The reason for this is that the Strichartz type norms available to us (lemma 3.1, lemma 6.7) are just not quite good enough for dealing with certain frequency interactions. Our way out of this is to exploit a (somewhat cumbersome) null-structure via further dynamic separations, and treat the expression similarly to the trilinear null-forms. Fortunately, we then don't have to take advantage of the same subtle cancellations as for the trilinear expressions. The septilinear error terms generated by this procedure are finally easy to estimate, essentially by means of Strichartz type norms.

3. Technical preparations

We shall employ Banach spaces closely modelled upon the ones in [33], [30], [18]. First, we recall the homogeneous Besov analogues of the classical $X^{s,b}$-spaces of Klainerman-Machedon: We introduce the Littlewood-Paley localizers P_k which restrict frequency to dyadic size $\sim 2^k$, $k \in \mathbf{Z}$. More precisely, choosing a smooth nonnegative bump function $m_0(x) : \mathbf{R} \to \mathbf{R}$ with support on $\frac{1}{4} < x < 4$ and satisfying

$$\sum_{k\in\mathbf{Z}} m_0(\frac{x}{2^k}) = 1,\, x \in \mathbf{R}_+,$$

we define for $f(x) \in \mathcal{S}(\mathbf{R}^2)$ or $f \in \mathcal{S}(\mathbf{R}^{2+1})$, $\widehat{P_k f}(\xi) := m_0(\frac{|\xi|}{2^k})\hat{f}(\xi)$[14]. Similarly, let Q_j, $j \in \mathbf{Z}$ microlocalize to dyadic distance $\sim 2^j$ from the light cone. Thus we put

$$\widetilde{Q_j\phi}(\tau,\,\xi) = m_0(\frac{|\,|\tau| - |\xi|\,|}{2^j})\tilde{\phi}(\tau,\xi).$$

We usually denote the (space-time) Fourier transform on $\mathbf{R}^{2+1}$ by $\tilde{}$, and use $(\tau,\,\xi)$ as coordinates on the (space-time) Fourier side. For future reference, we also introduce the multipliers $Q_j^{\pm}$, where

[14] More details are to be found in the fundamental work [23]

$$\widetilde{Q_j^{\pm}\phi}(\tau,\,\xi) = m_0(\frac{|\,|\tau| - |\xi|\,|}{2^j})\chi_{><0}(\tau)\tilde{\phi}(\tau, \xi),$$

and $\chi_{>0}$, $\chi_{<0}$ are, respectively, the Heaviside function localizing to the upper or lower half-space $\tau > 0$, $\tau < 0$. Observe that for Schwartz functions $\psi \in \mathcal{S}(\mathbf{R}^{2+1})$ we have

$$Q_j \psi = Q_j^+ \psi + Q_j^- \psi.$$

We also have the obviously defined variants $Q_{<>j}$ etc. The quantity 2^j will be called **modulation**, a notation inherited from [30]. Now we put[15]

$$||\phi||_{\dot{X}_k^{\lambda,p,q}} = 2^{\lambda k}(\sum_{j\in\mathbf{Z}}[2^{pj}||Q_j\phi||_{L_t^2 L_x^2}]^q)^{\frac{1}{q}}.$$

We shall mostly need the versions corresponding to the triples $(0, \frac{1}{2}, \infty)$ and $(0, \frac{1}{2}, 1)$. Furthermore, we introduce the Banach spaces $S[k,\kappa]$ as in [30], [18][16], where $k \in \mathbf{Z}$ indexes the frequency region, and $\kappa \subset S^1$ is a small cap. These spaces consist of three ingredients:

$$||\phi||_{S[k,\kappa]} = ||\phi||_{L_t^\infty L_x^2} + 2^{-\frac{k}{2}}|\kappa|^{-\frac{1}{2}}||\phi||_{PW[\kappa]} + ||\phi||_{NFA^*[\kappa]}.$$

Here we let $PW[\kappa]$ be the atomic Banach space whose atoms are Schwartz functions $\psi \in \mathcal{S}(\mathbf{R}^{2+1})$ with the property

$$\exists \omega \in \kappa \,\text{s.t.}\, ||\psi||_{L_{t_\omega}^2 L_{x_\omega}^\infty} \leq 1.$$

Also, we define

$$||\psi||_{NFA[\kappa]^*} = \sup_{\omega\notin 2\kappa}\text{dist}(\omega,\kappa)||\psi||_{L_{t_\omega}^\infty L_{x_\omega}^2}.$$

We immediately observe that these definitions imply the following **fundamental bilinear inequality**: assume $2\kappa \cap 2\kappa' =$. Then

$$||\phi\psi||_{L_t^2 L_x^2} \lesssim \frac{|\kappa'|^{\frac{1}{2}}2^{\frac{k'}{2}}}{\text{dist}(\kappa,\kappa')}||\phi||_{S[k,\kappa]}||\psi||_{S[k',\kappa']}. \tag{14}$$

Furthermore, as in [30], [18], we note the following fundamental relation between these new spaces and $X^{s,b}$ type spaces just introduced : let $\psi \in \mathcal{S}(\mathbf{R}^{2+1})$. Then

[15]For lots of information concerning $X^{s,\theta}$ spaces in the subcritical context, consult [14].

[16]We need to adjust the scaling properties, of course. These function spaces were in essence invented by D. Tataru.

we have

$$||P_kQ^{\pm}_{<k}\psi||_{S[k,\pm\kappa]}\lesssim||P_k\psi||_{\dot{X}_k^{0,\frac{1}{2},1}}. \tag{15}$$

Finally, the $S[k,\kappa]$ satisfy a crucial **orthogonality property**, see [33], [30] or also [18]: Let ψ_κ be a Schwartz function whose Fourier support is contained in an angular sector κ of radius r. Then, provided $\{\omega\}$ is a finitely overlapping cover of κ by caps ω of size 2^l, $l<\log_2 r-5$, and we have chosen operators $P_{0,\omega}$ smoothly microlocalizing to ω and satisfying $\sum_\omega P_{0,\omega}\psi_\kappa=P_0\psi_\kappa$, we have the inequality

$$||P_0Q^{+}_{<2l}\psi_\kappa||_{S[0,\kappa]}\lesssim(\sum_\omega||P_{0,\omega}Q^{+}_{<2l}\psi_\kappa||^2_{S[0,\omega]})^{\frac{1}{2}}. \tag{16}$$

With these ingredients in hand, we now construct the spaces $S[k]$, $k\in\mathbf{Z}$, which are to hold the k-th frequency component of $\psi=(\psi^i_\alpha)$: for each $l\in\mathbf{Z}$, $l<-10$, choose a finitely overlapping covering K_l of S^1 by caps κ of diameter 2^l, and such that these coverings are uniformly finitely overlapping in l. Moreover, for every such l, and $\lambda\in\mathbf{Z}$ with the property $-10\geq\lambda\geq l$, subdivide the angular sector $\{\xi\in\mathbf{R}^2|\frac{\xi}{|\xi|}\in\kappa,\,|\xi|\sim 2^k\}$ into a finitely overlapping (uniformly in l,λ) collection $C_{k,l,\lambda}$ of slabs R of width $2^{k+\lambda}$. Finally, let $0<M<\infty$ be a large positive number. Then we define

$$\begin{aligned}||\psi||_{S[k]}:=&||\psi||_{L^\infty_tL^2}+||\psi||_{\dot{X}_k^{0,\frac{1}{2},\infty}}+||\psi||_{\dot{X}_k^{-\frac{1}{2},1,2}}+||P_kQ_{\geq k}\partial_t\psi||_{L^M_t\dot{H}^{-1+\frac{1}{M}}}\\&+\sup_\pm\sup_{l<-10}\sup_{-10\geq\lambda\geq l}|\lambda|^{-1}(\sum_{\kappa\in K_l}\sum_{R\in C_{k,\kappa,\lambda}}||\tilde{P}_RQ^{\pm}_{<k+2l}\psi||^2_{S[k,\pm\kappa]})^{\frac{1}{2}}.\end{aligned} \tag{17}$$

The operator $\tilde{P}_R$ microlocalizes to the slab $R\in C_{k,l,\lambda}$ and is given by a multiplier $m_R(|\xi|)a_\kappa(\frac{\xi}{|\xi|})$, where $a_\kappa:S^1\to\mathbf{R}$ is a nonnegative smooth bump function supported on κ, and such that we have the properties

$$\sum_{\kappa\in K_l}a_\kappa|_{S^1}=1|_{S^1},\quad\sum_{R\in C_{k,\kappa,\lambda}}\tilde{P}_R=P_{k,\kappa},$$

the latter multiplier given by the symbol $m_0(\frac{|\xi|}{2^k})a_\kappa(\frac{\xi}{|\xi|})$. The following crude estimate follows immediately from the definition and the preceding remarks: let $\psi\in\mathcal{S}(\mathbf{R}^{2+1})$:

$$||P_0\psi||_{S[0]}\lesssim||P_0\psi||_{L^2_tL^2_x}+||P_0\partial_t\psi||_{L^2_tL^2_x}+||P_0\partial_t\psi||_{L^M_tL^2_x} \tag{18}$$

We also have

$$||P_0Q_{<0}\psi||_{S[0]}\lesssim||P_0\psi||_{\dot{X}_0^{0,\frac{1}{2},1}} \tag{19}$$

The definition above is somewhat hard to digest, of course. The nature of these spaces is really revealed by means of the **bilinear** estimates which they satisfy, one of the most pivotal of which is (14). Another manifestation of this 'bilinear character' is expressed by the following lemma, which is proved exactly as in [19], in the 3-dimensional context (see also section 6, lemma 6.7):

Lemma 3.1. *Let $p > 4$, $q > 0$ and $\frac{1}{p} + \frac{1}{2q} < \frac{1}{4}$. Then for any $\psi \in \mathcal{S}(\mathbf{R}^{2+1})$, we have the inequality*

$$||P_0\psi||_{L_t^p L_x^q} \lesssim ||P_0\psi||_{S[0]}$$

Finally, for our bootstrap argument, we need the time-localized versions of these spaces: given $T > 0$, we thus define (as in [30], [18])

$$||\psi||_{S[k]([-T,T]\times\mathbf{R}^2)} := \inf_{\tilde{\psi}|_{[-T,T]}=\psi|_{[-T,T]}} \{||\tilde{\psi}||_{S[k]}\}$$

Here both ψ, $\tilde{\psi}$ are Schwartz functions. Given the complexity of the spaces, it is not even clear that this family of norms depends continuously on the parameter T. For this, we state the following lemma, whose proof follows a suggestion by D. Tataru:

Lemma 3.2. *Let $\psi \in C^\infty([-T_0, T_0] \times \mathbf{R}^2)$, $T_0 > 0$. Then the norms*

$$||P_k\psi||_{S[k]([-T,T]\times\mathbf{R}^2)}$$

depend continuously on T for $T_0 > T \geq 0$.

Proof : Observe that the definition of $S[k]$ can be extended to non-integral values of k. Given $T > 0$, choose ϵ very small. Put $\frac{T+\epsilon}{T} = \lambda$. Then we have

$$||P_k\psi||_{S[k]([-T-\epsilon,T+\epsilon]\times\mathbf{R}^2)} = \lambda||P_{k+\log_2\lambda}\psi_\lambda||_{S[k+\lambda]([-T,T]\times\mathbf{R}^2)}$$

where we have put $\psi_\lambda(t,x) = \psi(\lambda t, \lambda x)$. Now we estimate

$$\begin{aligned}
&|\ ||P_k\psi||_{S[k]([-T-\epsilon,T+\epsilon]\times\mathbf{R}^2)} - ||P_k\psi||_{S[k]([-T,T]\times\mathbf{R}^2)}\ | \\
&\leq |\ ||P_k\psi||_{S[k]([-T,T]\times\mathbf{R}^2)} - ||P_{k+\log_2\lambda}\psi_\lambda||_{S[k+\log_2\lambda]([-T,T]\times\mathbf{R}^2)}\ | \\
&\qquad\qquad + |\lambda - 1|||P_{k+\log_2\lambda}\psi_\lambda||_{S[k+\log_2\lambda]([-T,T]\times\mathbf{R}^2)} \\
&\lesssim ||P_k\psi - P_{k+\log_2\lambda}\psi_\lambda[0]||_{L_x^2} + ||\Box(P_k\psi - P_{k+\log_2\lambda}\psi_\lambda)||_{L_t^1\dot{H}^{-1}} \\
&\qquad\qquad + |\lambda - 1|||P_{k+\log_2\lambda}\psi_\lambda||_{S[k+\log_2\lambda]([-T,T]\times\mathbf{R}^2)}
\end{aligned}$$

Letting $\epsilon \to 0$ whence $\lambda \to 1$ yields the claim. In the last inequality, we have used the 'energy inequality' (21) to be discussed below. Continuity at $T = 0$ follows directly from the 'energy inequality'. ■

As insinuated in the first section, we shall not be able to place the (frequency localized) nonlinearity into the classical energy space $L_t^1\dot{H}^{-1}$ (which has the right scaling). We shall use spaces $N[k]$, $k \in \mathbf{Z}$, as well as time localized versions $N[k]([-T,T]\times\mathbf{R}^2)$ for that purpose, which, up to minor wrinkles, are defined in perfect analogy with [30], [18]: indeed, we let $N[k]$ be the atomic Banach space whose atoms are Schwartz functions $F\in\mathcal{S}(\mathbf{R}^{2+1})$, at frequency $\sim 2^k$, as well as satisfying one of the following properties:

(1) $||F||_{L_t^1\dot{H}^{-1}}\leq 1$ and F has modulation $<2^{k+100}$.

(2) F is at modulation $\sim 2^j$ and satisfies $||F||_{L_t^2L_x^2}\leq 2^{\frac{j}{2}}2^k$.

(3) F satisfies $||F||_{\dot{X}_k^{-\frac{1}{2},-1,2}}\leq 1$, and one can write $F=\partial_tF'$ for some $F'\in\mathcal{S}(\mathbf{R}^{2+1})$ with the property $||F'||_{L_t^ML_x^2}\leq 2^{(1-\frac{1}{M})k}$, for M as in the definition of $S[k]$.

(4) There exists an integer $l<-10$, and Schwartz functions F_κ with Fourier support in the region

$$\{(\tau,\xi)|\pm\tau>0,\,|\,|\tau|-|\xi|\,|\leq 2^{k-2l-100},\,\frac{\xi}{|\xi|}\in\pm\kappa\}$$

with the properties

$$F=\sum_{\kappa\in K_l}F_\kappa,\,(\sum_{\kappa\in K_l}||F_\kappa||^2_{NFA[\kappa]})^{\frac{1}{2}}\leq 2^k$$

In the last inequality, $NFA[\kappa]$ denotes the dual of $NFA[\kappa]^*$ used in the definition of $S[k,\kappa]$.

We immediately note the 2nd pivotal bilinear inequality, which is essentially dual to (14): letting the notation and assumptions be like there, we have

$$||\phi\psi||_{NFA[\kappa]}\lesssim\frac{2^{\frac{k'}{2}}|\kappa'|^{\frac{1}{2}}}{\mathrm{dist}(\kappa,\kappa')}||\phi||_{L_t^2L_x^2}||\psi||_{S[k',\kappa']}\tag{20}$$

Note that $NFA[\kappa]$ is the atomic Banach space whose atoms are Schwartz functions F satisfying

$$\frac{1}{\mathrm{dist}(\omega,\kappa)}||F||_{L^1_{t_\omega}L^2_{x_\omega}}\leq 1$$

for some $\omega\notin 2\kappa$. We still need to tie the two classes of spaces $N[k]$, $S[k]$ or their time-localized versions together by means of an **energy inequality**. This shall be proved in an appendix, essentially as in [33], [30], and can be stated as follows[17]:

[17]For the proof of lemma 3.2, we note that there is similar 'energy inequality' without the factor $\min\{2^kT_0,1\}^{-\frac{1}{M}}$ provided $N[k]$ is replaced by $L_t^1\dot{H}^{-1}$.

$$||P_k\psi||_{S[k]([-T,T]\times\mathbf{R}^2)} \lesssim \inf_{0<T_0\leq T}[\min\{2^kT_0,1\}^{-\frac{1}{M}}||\Box P_k\psi||_{N[k]([-T,T]\times\mathbf{R}^2)} + \sup_{t_0\in[-T_0,T_0]}||P_k\psi[t_0]||_{L^2\times\dot{H}^{-1}}]. \quad (21)$$

The factor $\min\{2^kT_0,1\}^{-\frac{1}{M}}$ is a technical nuisance due to the fact that time derivatives are not controlled on individual time-slices, but only when 'averaged out'. As we can restrict ourselves to $k=0$, it only plays a role for small T_0. The very difficult fact that an inequality of this type holds for the kind of spaces considered here is due originally to D. Tataru, but we shall mostly rely on a proof given by Tao [30]. We close this section with two important inequalities which shall be used constantly in the sequel. The first is the **classical Bernstein's inequality**, which states that for $\psi\in\mathcal{S}(\mathbf{R}^n)$, and R a measurable set, we have

$$||\mathcal{F}^{-1}(\chi_R\mathcal{F}(\psi))||_{L_t^2L_x^p} \lesssim |R|^{\frac{1}{2}-\frac{1}{p}}||\psi||_{L_t^2L_x^2}.$$

Next, the **improved Bernstein's inequality** of Tao [30], which is a Strichartz type inequality in disguise, states that with the same notation and $n\geq 2$:

$$||P_kQ_j\psi||_{L_t^2L_x^\infty} \lesssim 2^{\frac{kn}{2}}2^{\frac{j-k}{4}}||P_k\psi||_{L_t^2L_x^2}. \quad (22)$$

A[18] dual version of this is

$$||P_kQ_j\psi||_{L_t^2L_x^2} \lesssim 2^{\frac{kn}{2}}2^{\frac{j-k}{4}}||P_k\psi||_{L_t^2L_x^1}.$$

Notational and semantic idiosyncrasies: We shall frequently have to consider nested expressions of the schematic form

$$[\psi_1\nabla^{-1}(\psi_2\nabla^{-1}[\psi_3\psi_4])].$$

In these ∇^{-1} always refers to a linear combination of operators $\triangle^{-1}\partial_k$, $k=1,2$, which act on frequency localized (Schwartz) functions in the obvious way[19]. We shall refer to ψ_i, $i=1,\ldots 4$ as **inputs** and the whole expression as **output**[20]. Moreover, we shall frequently use the operator $I=\sum_{k\in\mathbf{Z}}P_kQ_{<k+10}$ or obvious variations. In an expression $(\psi_1 I\psi_2\psi_3 ...)$, it is understood that I acts only on the immediately following input ψ_2. Our strategy for estimating a *null-form expression* shall often consist in first *reducing its inputs as well as the output to 'hyperbolic microsupport'*. For example, when we say *"reduce $P_{k_1}\psi_1$ to modulation $<2^\alpha$ in the expression $[P_{k_1}\psi_1P_{k_2}\psi_2\ldots]$"*, this means we estimate the expression $[P_{k_1}Q_{\geq\alpha}\psi_1P_{k_2}\psi_2\ldots]$. Having achieved this estimation allows us to restrict ourselves to estimating $[P_{k_1}Q_{<\alpha}\psi_1P_{k_2}\psi_2\ldots]$. In order to keep the expressions manageable, we shall frequently omit subscripts and indices. The meaning will be clear

[18]The inequality is only optimal for $n=2$, the case we need here. See for example [18].
[19]Thus ∇^{-1} only acts on the spatial variables!
[20]We adopt this useful terminology from Tao's work [30].

from the context. Finally, we shall frequently use the following observation: let $k_1 >> k_2$, and consider the expression

$$\nabla^{-1}(P_{k_1}\psi_1 P_{k_2}\psi_2) = \nabla^{-1}P_{k_1+O(1)}(P_{k_1}\psi_1 P_{k_2}\psi_2).$$

In the last expression the operator $\nabla^{-1}P_{k_1+O(1)}$ is given by convolution with a smooth kernel $a(.) \in \mathcal{S}(\mathbf{R}^2)$ of L^1-mass $\sim 2^{-k_1}$, which means we can rewrite the preceding, evaluated at $(t,x) \in \mathbf{R}^{2+1}$ as

$$\int_{\mathbf{R}^2} a(y) P_{k_1} T_y \psi_1(t,x) P_{k_2} T_y \psi_2(t,x) dy.$$

In the preceding formula T_y is the translation operator $T_y f(.) = f(. - y)$. We also use the fact that translation and localization on the Fourier side commute. When estimating an expression containing $\nabla^{-1}(P_{k_1}\psi_1 P_{k_2}\psi_2)$, we can think of this expression as a superposition of expressions arising for fixed $y \in \mathbf{R}^2$. Using the translation invariance of all Banach spaces as well as the triangle inequality, we see that as far as estimates are concerned, we may replace $\nabla^{-1}(P_{k_1}\psi_1 P_{k_2}\psi_2)$ by $(\nabla^{-1}P_{k_1}\psi_1 P_{k_2}\psi_2)$ under the preceding assumptions on the frequencies. Simple variations of this kind of reasoning shall be ubiquitous throughout the paper. For example, we sometimes use the fact that operators of the form $P_k Q_{<>k+O(1)}$ are given by convolution with an L^1-bounded kernel in space-time, see e. g. [30].

4. Preparing the bootstrapping: Part I

Following [29], [30], we introduce Tao's concept of *frequency envelope*: we denote a sequence of nonnegative numbers $\{c_k\}_{k\in\mathbf{Z}}$ a frequency envelope provided there exists a positive number $\sigma > 0$ with the property

$$c_a 2^{-\sigma|a-b|} \leq c_b \leq c_a 2^{\sigma|a-b|}.$$

Of particular relevance for us is the following kind of frequency envelope: take the initial data $\psi(0) = (\psi_\alpha(0))$, $\sigma > 0$, and form

$$c_k = (\sum_{l\in\mathbf{Z}} 2^{-\sigma|k-l|} ||P_l\psi(0)||_{L_x^2}^2)^{\frac{1}{2}}.$$

The following Proposition is the heart of the paper:

Proposition 4.1. *Let $\psi := \{\psi_\nu\}$ be as in the preceding discussion. Given $K > 0$ sufficiently large, there exist $\sigma > 0$, $\epsilon > 0$ sufficiently small, such that the following holds: assume that for some $T > 0$, we have*

$$||P_k\psi||_{S[k]([-T,T]\times\mathbf{R}^2)} \leq K c_k,\ ||\psi(0)||_{L_x^2} \leq \epsilon.$$

Then, the first inequality holds with $\frac{K}{2}$ instead.

Proposition 4.1 implies Theorem 1.1: Finite speed of propagation allows us to assume that the initial data are compactly supported. Assume that $(-T,T)$, $T>0$ is a maximal interval of existence for the Wave Map. Lemma 3.2 and rapid decay of the Fourier transformed components ψ_ν on any time slice in $(-T,T)$ implies that for any $K>0$ sufficiently large, there exists some $T'\in(0,T)$ with the property

$$\sup_{k\in\mathbf{Z}} c_k^{-1}||P_k\psi||_{S[k]([-T',T']\times\mathbf{R}^2)}=K$$

But this contradicts Proposition 4.1, so we conclude that there is some $K_0>0$ with the property $||P_k\psi||_{S[k]([T',T']\times\mathbf{R}^2)}\leq K_0c_k\,\forall T'\in[0,T)$. From the definition of $\{c_k\}$, we infer that $||\psi||_{L_t^\infty H^\delta([-T,T]\times\mathbf{R}^2)}<\infty$ for $0\leq\delta<\sigma$, where σ is as in the Proposition. Choose $\delta_{1,2}$ with $\sigma>\delta_1>\delta_2>0$. On every fixed time slice $t=\text{const}$, $t\in(-T,T)$, we compute

$$\begin{aligned}
||P_l\phi_\nu||_{H^{\delta_2}} &= ||P_l[e^{i\sum_j\triangle^{-1}\partial_j\phi_j^1}\psi_\nu]||_{H^{\delta_2}}\\
&\leq ||P_l[P_{<l-10}(e^{i\sum_j\triangle^{-1}\partial_j\phi_j^1})P_{[l-10,l+10]}\psi_\nu]||_{H^{\delta_2}}\\
&\quad + ||P_l[P_{[l-10,l+10]}(e^{i\sum_j\triangle^{-1}\partial_j\phi_j^1})P_{<l+15}\psi_\nu]||_{H^{\delta_2}}\\
&\quad + \sum_{k>l+10}||P_l[P_k(e^{i\sum_j\triangle^{-1}\partial_j\phi_j^1})P_{k+O(1)}\psi_\nu]||_{H^{\delta_2}}\\
&\lesssim ||P_{[l-10,l+10]}\psi_\nu||_{H^{\delta_2}}+||P_{[l-10,l+10]}(e^{i\sum_j\triangle^{-1}\partial_j\phi_j^1})||_{H^{\delta_2}}||P_{<l+15}\psi_\nu||_{L^\infty}\\
&\quad + \sum_{k>l+10}||P_k(e^{i\sum_j\triangle^{-1}\partial_j\phi_j^1})||_{H^{\delta_2}}||P_{k+O(1)}\psi_\nu||_{L^\infty}
\end{aligned}$$

Furthermore, we have the inequalities

$$||\nabla_x e^{i\sum_j\triangle^{-1}\partial_j\phi_j^1}||_{L_x^2}\lesssim||\phi||_{L^2},\ ||P_{<l+15}\psi||_{L_x^\infty}\lesssim 2^{(1-\delta_1)l}||\psi||_{H^{\delta_1}}$$

whence we conclude

$$||P_l\phi_\nu||_{H^{\delta_2}}\lesssim||P_{l+O(1)}\psi||_{H^{\delta_2}}+2^{(\delta_2-\delta_1)l}||\psi||_{L^2}||\psi||_{H^{\delta_1}}$$

Square summing over $l>0$, we obtain a global bound on $||\phi||_{H^{\delta_2}}$. By the subcritical result of Klainerman-Machedon, we can continue the Wave Map beyond $[-T,T]$, which contradicts the assumption.

We now commence with the proof of Proposition 4.1, which will occupy the rest of the paper. By scaling invariance, we reduce to bootstrapping a single frequency component $||P_0\psi_\nu||_{S[0]([-T,T]\times\mathbf{R}^2)}$, $\nu=0,1,2$. We first dispose of the case for small T: $T_0\geq T>0$ where T_0 remains to be chosen. Observe that the divergence-curl system and in particular (6) implies

$$P_0\psi_i(t,.)=P_0\psi_i(0,.)+\int_0^t P_0[\partial_i\psi_t](s,.)ds+\int_0^t P_0[\psi\nabla^{-1}(\psi^2)](s,.)ds$$

where the 2nd integrand is of course written schematically, $i=1,2$. We observe that lemma 3.1 and the assumptions in Proposition 4.1 imply

$$||P_0[\psi\nabla^{-1}(\psi^2)]||_{L_t^M L_x^2}\leq CK^3c_0(\sum_{k\in\mathbf{Z}}c_k^2)$$

Using Hoelder's inequality, we deduce that

$$||P_0\psi_i(t,.)||_{L^2}\lesssim c_0(1+KT+K^3T^{1-\frac{1}{M}})$$

Choosing K large enough and T_0 small enough, we infer that

$$||P_0\psi_i||_{L_t^\infty L_x^2([-T,T]\times\mathbf{R}^2)}<\frac{K}{100}c_0,$$

provided $T\in[0,T_0]$. Again using Hoelder, one gets the same bound for $||P_0\psi_i||_{L_t^2 L_x^2([-T,T]\times\mathbf{R}^2)}$. Similarly, using the divergence-curl system, the same conclusion follows for $||P_0\partial_t\psi_i||_{L_t^2L_x^2}$, $||P_0\partial_t\psi_i||_{L_t^M L_x^2}$, provided we also choose $\sum_{k\in\mathbf{Z}}c_k^2$ small enough. Now we build a Schwartz extension of $P_0\psi_i|_{[T,T]}$ as follows: first choose a Schwartz function $f_i(t,x)$ extending $P_0\partial_t\psi_i|_{[T,T]}$ as well as satisfying

$$||P_0f_i||_{L_t^M L_x^2}+||P_0f_i||_{L_t^2L_x^2}<\frac{K}{100}c_0$$

which is possible according to the preceding considerations. Then we set

$$\tilde{\psi}_i:=\eta_{T_1}(t)[P_0\psi_i(0,x)+\int_0^t f_i(s,x)ds]$$

where $\eta_{T_1}, T_1>T_0$, is a smooth cutoff supported in $[-2T_1,2T_1]$ with $\eta_{T_1}|_{[-T_1,T_1]}\equiv 1$ and $||\partial_t\eta_{T_1}(t)||_{L_t^\infty}\leq 2T_1^{-1}$. Observe that

$$||\partial_t\eta_{T_1}(t)\int_0^t f_i(s,x)ds||_{L_t^\infty L_x^2}\leq\frac{K}{50}T_1^{-1}c_0$$

This yields that

$$||\partial_t\eta_{T_1}(t)\int_0^t f_i(s,x)ds||_{L_t^M L_x^2}+||\partial_t\eta_{T_1}(t)\int_0^t f_i(s,x)ds||_{L_t^2L_x^2}\leq\frac{K}{10}c_0$$

The desired properties of $\tilde{\psi}_i$ follow from this if necessary replacing 10 by a bigger factor to counteract the loss in (18). The claim for ψ_0 follows similarly upon invoking (7).

5. Preparing the bootstrapping; Part II

5.1. **Preliminaries.** Now we assume that the Wave Map exists on a time interval $[-T,T]$ where $T \geq T_0$, the latter as in the previous section. We shall now revert to the wave equations satisfied by the ψ_α. More precisely, writing schematically

$$\Box\psi_\alpha = F_\alpha,$$

where F_α stands for the expression in (8), we need to choose a Schwartz extension $\tilde{F}_\alpha$ of $F_\alpha|_{[-T,T]}$, and solve the corresponding wave equation with given Cauchy data. An appropriate truncation will then define our new Schwartz extension of $\psi_\alpha|_{[-T,T]}$. A good candidate for $\tilde{F}_\alpha$ is of course obtained by simply substituting suitable Schwartz extensions of ψ_α implied by the bootstrap assumptions in the Proposition. However, this will not result in good terms. Another option is to apply dynamic separation, yielding trilinear null-forms and quintilinear error terms. Unfortunately, it turns out that the trilinear null-forms cause trouble in certain elliptic regimes, and we have to apply a somewhat messy 'partial dynamic separation' in which not all terms are decomposed into hyperbolic and elliptic parts, depending on microlocal properties of other terms. Leaving the details until later, we assume that we have found a suitable extension $\tilde{F}_\alpha$, $\alpha = 0,1,2$, for which the required estimates hold. In order to avoid confusion, we denote the putative extension of $P_0\psi_\alpha$ by ρ_α. Then we write

$$\Box\rho_\alpha = P_0 Q_{<0}\tilde{F}_\alpha + P_0 Q_{\geq 0}\tilde{F}_\alpha$$

We could write ρ_α as the sum of solutions of the inhomogeneous problems

$$\Box\rho_\alpha^1 = P_0 Q_{<0}\tilde{F}_\alpha,\ \Box\rho_\alpha^2 = P_0 Q_{\geq 0}\tilde{F}_\alpha,$$

with trivial initial data, as well as a free wave matching initial conditions, and finally truncate suitably. Unfortunately, this doesn't quite work, since control[21] over the latter would require control over the time derivatives $\partial_t\psi_\alpha(0)$, which doesn't quite follow from our assumptions, even using the divergence-curl system. We proceed instead as follows: Solve the elliptic problem by the formula

$$\rho_\alpha^2 = P_0 Q_{\geq 0}\Box^{-1}\tilde{F}_\alpha,$$

where $\Box^{-1}$ is given by the symbol $(|\tau|^2 - |\xi|^2)^{-1}$ on the (space-time) Fourier side. We shall place $P_0 Q_{\geq 0}\tilde{F}_\alpha$ into the space

$$\dot{X}_0^{-\frac{1}{2},-1,2} \cap \partial_t(L_t^M \dot{H}^{-(1-\frac{1}{M})}).$$

In particular, we can bound $||\partial_t\rho_\alpha^2||_{L_t^M L_x^2}$. Similarly, we can bound

[21]In the sense that one needs to control the frequency blocks in terms of the frequency envelope.

$||P_0[\psi\nabla^{-1}(\psi^2)]||_{L_t^M L_x^2}$, where the parenthesis stands for any of the expressions occuring in the divergence-curl system. We infer that there exists a slice $t = t_0 \in [-T_0, T_0]$ for which an inequality of the form

$$||\partial_t \rho_\alpha^2||_{L_x^2} + ||\partial_t P_0\psi_\alpha||_{L_x^2} + ||P_0\psi_\alpha||_{L_x^2} + ||P_0\rho_\alpha||_{L_x^2} \lesssim c_0 + K^3 c_0 \sum_{k\in\mathbf{Z}} c_k^2$$

holds true. Now we solve the inhomogeneous hyperbolic problem

$$\Box\rho_\alpha^1 = P_0 Q_{<0}\tilde{F}_\alpha,$$

with trivial conditions at time t_0, and finally solve the free wave equation

$$\Box\rho_\alpha^3 = 0,\ \rho_\alpha^3[t_0] = P_0\psi_\alpha[t_0] - P_0\rho_\alpha^2[t_0]$$

Truncating suitably, we can define the new extension as $\psi_\alpha = \sum_{i=1}^3 \rho_\alpha^i$.

5.2. **Defining the extension of the nonlinearity.** We now have to define $\tilde{F}_\alpha$. Enacting dynamic separation in the innermost square brackets $[,]$ on the right hand side of (8) and only considering the resulting trilinear expressions, we obtain embedded $Q_{\nu j}$ null-forms applied to suitable (linear combinations of) terms ψ_α, where we define

$$Q_{\nu j}(u,\, v) = R_\nu u R_j v - R_j u R_\nu v$$

We substitute the Schwartz extensions for the locally defined ψ_α as in the assumptions of Proposition 4.1. We shall then apply dynamic separation or not to the first factor in the nonlinearity depending on whether the innermost bracket with inputs as just specified has microsupport close to the light cone or not. Then we substitute suitable extensions for this factor also. In symbols, defining the operator $I = \sum_{k\in\mathbf{Z}} P_k Q_{<k+10}$ and letting $\tilde{\psi_\alpha}$ be suitable Schwartz extensions of ψ_α, our first candidate for $\tilde{F}_\alpha$ is the following:

$$\begin{aligned}
&\tilde{F}_\alpha = i\partial^\beta[R_\alpha\tilde{\psi}\triangle^{-1}\sum_{j=1}^2 \partial_j I[R_\beta\tilde{\psi}^1 R_j\tilde{\psi}^2 - R_\beta\tilde{\psi}^2 R_j\tilde{\psi}^1]]\\
&+ i\partial^\beta[\tilde{\psi_\alpha}\triangle^{-1}\sum_{j=1}^2 \partial_j(1-I)[R_\beta\tilde{\psi}^1 R_j\tilde{\psi}^2 - R_\beta\tilde{\psi}^2 R_j\tilde{\psi}^1]]\\
&+ \text{similar trilinear terms}\\
&+ \nabla_{x,t}[\nabla^{-1}(\tilde{\psi}\nabla^{-1}(\tilde{\psi}^2))\nabla^{-1} I Q_{\nu j}(\tilde{\psi},\tilde{\psi})] + \nabla_{x,t}[\tilde{\psi}\nabla^{-1}(\nabla^{-1}(\tilde{\psi}\nabla^{-1}(\tilde{\psi}^2))R_\beta\tilde{\psi})],\\
&+ \text{higher order terms}
\end{aligned}$$

where we have put $-\sum_{k=1,2} R_k\tilde{\psi}_k^{1,2} = \tilde{\psi}^{1,2}$ as well as $\tilde{\psi} = \tilde{\psi}^1 + i\tilde{\psi}^2$, and the quintilinear terms are only recorded schematically. As alluded to in the first section, the

2nd quintilinear term here isn't quite good enough. We apply dynamic separation to the innermost curly bracket ($\tilde{\psi}^2$), replacing it by a $Q_{\nu j}$-type null-form as well as error terms at least quadrilinear. Focusing on the quintilinear term thus generated and substituting appropriate inputs, we then distinguish between the case when $Q_{\nu j}(\tilde{\psi}, \tilde{\psi})$ is microlocalized to the 'elliptic region' or the 'hyperbolic region', as in the preceding discussion. In the latter case, further dynamic separations need to be applied to the remaining inputs, resulting in a complicated quintilinear null-form and a slew of error terms at least septilinear. The latter will be much easier to estimate. Our main focus in the immediately following shall be on the most difficult **trilinear null-forms**, leaving the quintilinear and higher order terms to later sections.

6. Bi- and trilinear estimates

6.1. **Bilinear estimates.** This subsection is preparatory. The main trilinear estimates start in the next subsection. We proceed in close analogy to [19]. The first estimate we prove here is the strengthened[22] $2+1$-dimensional analogue of a corresponding estimate in [19]. It is the same as the analogous estimate for free waves, see [8]:

Lemma 6.2. *Let $\psi_{1,2} \in \mathcal{S}(\mathbf{R}^{2+1})$. Then, for any $0 \leq p < \frac{1}{4}$, we have*[23]

$$||P_k[R_1 P_{k_1}\psi_1 R_2 P_{k_2}\psi_2 - R_2 P_{k_1}\psi_1 R_2 P_{k_2}\psi_2]||_{\dot{X}_0^{p,-p,2}} \leq C_p 2^{\frac{\min\{k_1,k_2,k\}}{2}} \prod_{i=1,2} ||P_{k_i}\psi_i||_{S[k_i]} \tag{23}$$

The same inequality holds if one of $R_{1,2}$ is replaced by R_0, provided the ψ_i are microlocalized to different half-spaces $\tau >< 0$, or one applies the operator I from the previous section in front of the expression, or else provided one includes an extra factor of the form $|\max\{k_1 - k,\, k_2 - k\}|^2$ on the right-hand side. In particular, we have an inequality of the form

$$||P_k[R_1\psi_1 R_2\psi_2 - R_2\psi_1 R_1\psi_2]||_{L_t^2 L_x^2} \lesssim (\sum_{k\in\mathbf{Z}} ||P_k\psi_1||_{S[k]}^2)^{\frac{1}{2}} (\sum_{k\in\mathbf{Z}} ||P_k\psi_1||_{S[k]}^2)^{\frac{1}{2}}$$

Proof : We prove the first inequality provided $R_{1,2}$ are replaced by $R_{0,1}$, in the **high-high interaction case**. The other cases are handled analogously and are simpler. Thus we assume $k_1 = k_2 + O(1) \geq 100$. The estimate is proved by considering various cases. By scale-invariance, we may suppose $k = 0$:

(1): One input at modulation $\geq 2^{k_1-100}$, i. e. the estimate provided we replace $P_{k_1}\psi_1$ by $P_{k_1}Q_{\geq k_1-100}\psi_1$: the null-structure is useless. Freeze the output to dyadic

[22] This statement follows a suggestion of D. Tataru, and clarifies an earlier weaker version by the author used at this point. We will only need the $L_t^2 L_x^2$-estimate, though.

[23] Recall that $R_\nu = (\sqrt{-\triangle_x})^{-1}\partial_\nu$.

modulation $\sim 2^j$. We estimate

$$\begin{aligned}&||P_0Q_j[P_{k_1}Q_{\geq k_1-100}R_0\psi_1P_{k_2}R_1\psi_2]||_{L_t^2L_x^2}\\ &\lesssim 2^{\min\{\frac{j}{4},0\}}||P_{k_1}Q_{\geq k_1-100}R_0\psi_1||_{L_t^2L_x^2}||P_{k_2}R_1\psi_2||_{L_t^\infty L_x^2}\\ &\lesssim 2^{\min\{\frac{j}{4},0\}}||P_{k_1}\psi_1||_{S[k_1]}||P_{k_2}\psi_2||_{S[k_2]}.\end{aligned}$$

We have used the dual of the improved Bernstein's inequality. The desired estimate follows easily from this, upon multiplying by 2^{-pj} and square-summing over j, using Plancherel's theorem.

(2): Both inputs microlocalized closely to the light cone, i. e. include an operator $Q_{<k_1-100}$ in front of each. First, decomposing

$$P_{k_1}Q_{<k_1-100}\psi_1 = P_{k_1}Q^+_{<k_1-100}\psi_1 + P_{k_1}Q^-_{<k_1-100}\psi_1,$$

distinguish between the cases when the inputs are microlocalized to the same or opposite half-spaces. In the former case, observe that we have an identity of the form

$$\begin{aligned}&P_0[P_{k_1}Q^\pm_{<k_1-100}\psi_1P_{k_2}Q^\pm_{<k_1-100}\psi_2]\\ &= \sum_{\kappa_{1,2}\in K_{-50},\, \mathrm{dist}(\kappa_1,-\kappa_2)\lesssim 2^{-50}} P_0[P_{k_1,\kappa_1}Q^\pm_{<k_1-100}\psi_1P_{k_2,\kappa_2}Q^\pm_{<k_1-100}\psi_2]\end{aligned}$$

where we recall the notation of section 3. One easily checks that

$$\begin{aligned}&P_0[P_{k_1,\kappa_1}Q^\pm_{<k_1-100}\psi_1P_{k_2,\kappa_2}Q^\pm_{<k_1-100}\psi_2]\\ &\qquad\qquad = P_0Q_{>50}[P_{k_1,\kappa_1}Q^\pm_{<k_1-100}\psi_1P_{k_2,\kappa_2}Q^\pm_{<k_1-100}\psi_2],\end{aligned}$$

hence throwing in the operator I in front of it will kill this term. Otherwise, the null-form is again useless, and we estimate by means of (14): use the more concise identity

$$\begin{aligned}&P_0[P_{k_1}Q^\pm_{<k_1-100}R_0\psi_1P_{k_2}Q^\pm_{<k_1-100}R_1\psi_2\\ &= \sum_{\kappa_{1,2}\in K_{-k_1-10},\, \mathrm{dist}(\kappa_1,-\kappa_2)\lesssim 2^{-k_1}} P_0[P_{k_1,\kappa_1}Q^\pm_{<k_1-100}R_0\psi_1P_{k_2,\kappa_2}Q^\pm_{<k_1-100}R_1\psi_2],\end{aligned}$$

whence, invoking (14) as well as Cauchy-Schwartz:

$$\begin{aligned}&||P_0[P_{k_1}Q^\pm_{<k_1-100}R_0\psi_1P_{k_2}Q^\pm_{<k_1-100}R_1\psi_2]||_{L_t^2L_x^2}\\ &\lesssim (\sum_{\kappa_1\in K_{-k_1-10}}||P_{k_1,\kappa_1}Q^\pm_{<k_1-100}\psi_1||^2_{S[k_1,\kappa_1]})^{\frac12}\\ &\qquad\qquad (\sum_{\kappa_2\in K_{-k_1-10}}||P_{k_2,\kappa_2}Q^\pm_{<k_1-100}\psi_2||^2_{S[k_2,\kappa_2]})^{\frac12}\end{aligned}$$

Using (15) as well as the definition of $S[k]$, one majorizes this by the right-hand side of (23) with an extra factor $|k_1|^2$.
We thus proceed to the case in which both inputs $P_{k_{1,2}}Q_{<k_1-100}\psi_{1,2}$ are microlocalized to opposite half-spaces $\tau >< 0$. In particular, we have the identity

$$\begin{aligned}
&R_0 P_{k_1} Q^+_{<k_1-100}\psi_1 R_1 P_{k_2} Q^-_{<k_1-100}\psi_2 - R_1 P_{k_1} Q^+_{<k_1-100}\psi_1 R_0 P_{k_2} Q^-_{<k_1-100}\psi_2\\
&= (R_0-1) P_{k_1} Q^+_{<k_1-100}\psi_1 R_1 P_{k_2} Q^-_{<k_1-100}\psi_2\\
&\qquad\qquad\qquad\qquad - R_1 P_{k_1} Q^+_{<k_1-100}\psi_1 (R_0+1) P_{k_2} Q^-_{<k_1-100}\psi_2\\
&+ P_{k_1} Q^+_{<k_1-100}\psi_1 R_1 P_{k_2} Q^-_{<k_1-100}\psi_2 + R_1 P_{k_1} Q^+_{<k_1-100}\psi_1 P_{k_2} Q^-_{<k_1-100}\psi_2
\end{aligned}$$

It is now entirely straightforward to estimate the first two summands, using the $\dot{X}_k^{0,\frac{1}{2},\infty}$-component of $S[k]$: for example

$$\begin{aligned}
&||P_0 Q_j[R_0-1) P_{k_1} Q^+_{<k_1-100}\psi_1 R_1 P_{k_2} Q^-_{<k_1-100}\psi_2||_{L_t^2 L_x^2}\\
&\lesssim 2^{\min\{\frac{j}{4},0\}}||R_0-1) P_{k_1} Q^+_{<k_1-100}\psi_1||_{L_t^2 L_x^2}||P_{k_2} Q^-_{<k_1-100}\psi_2||_{L_t^\infty L_x^2}\\
&\lesssim 2^{\min\{\frac{j}{4},0\}} 2^{-\frac{k_1}{2}}||P_{k_1}\psi_1||_{\dot{X}_0^{0,\frac{1}{2},\infty}}||P_{k_2}\psi_2||_{L_t^\infty L_x^2},
\end{aligned}$$

and one concludes as before. We are left with the remaining two summands. We now use a Whitney type decomposition as follows: for arbitrary functions $f,\, g \in \mathcal{S}(\mathbf{R}^{2+1})$ at large frequency $\sim 2^{k_1},\, 2^{k_2}$, respectively and a very large M to be chosen, we put

$$\begin{aligned}
P_0[fg] = &\sum_{\pm,\pm}\ \sum_{\omega_1,\tilde{\omega}_1\in K_{-k_1-100},\, 2^{-k_1+10}>\mathrm{dist}(\pm\omega_1,\pm\tilde{\omega}_1)\geq 2^{-k_1-90}} P_0[P_{k_1,\omega_1}f^\pm P_{k_2,\tilde{\omega}_1}g^\pm] +\\
&\sum_{\pm,\pm}\ \sum_{M\geq a>1}\ {\sum_{i\leq k,\, \omega_i,\tilde{\omega}_i\in K_{-90-k_1-10i}}}' P_0[P_{\omega_1}P_{\omega_2}\dots P_{\omega_a}f^\pm P_{\tilde{\omega}_1}P_{\tilde{\omega}_2}\dots P_{\tilde{\omega}_a}g^\pm]\\
&+\sum_{\pm,\pm}\ {\sum_{\omega_M,\tilde{\omega}_M\in K_{-90-k_1-10M}}}'' P_0[P_{\omega_1}P_{\omega_2}\dots P_{\omega_M}f^\pm P_{\tilde{\omega}_1}P_{\tilde{\omega}_2}\dots P_{\tilde{\omega}_M}g^\pm].
\end{aligned}$$

We let $f^\pm$, $g^\pm$ refer to the restrictions to the upper/lower half-space on the Fourier side. The sum $\sum'$ is carried out only over those pairs of caps $(\omega_i, \tilde{\omega}_i)$ which satisfy

$$2^{-80-k_1-10i} > \mathrm{dist}(\pm\omega_i, \pm\tilde{\omega}_i), 1\leq i < a-1,\, \mathrm{dist}(\pm\omega_a, \pm\tilde{\omega}_a)\geq 2^{-80-k_1-10a},$$

while in $\sum''$, the last $\geq$ above is replaced by $<$ for $a = M$. Now fix one of the summands in $\sum'$ or the first sum, i. e. choose a, $1\leq a < M$. We substitute $P_{k_1}Q^+_{<k_1-100}\psi_1$, $R_1 P_{k_2} Q^-_{<k_1-100}\psi_2$ for $f^\pm$, $g^\pm$, respectively. For each $\omega_a \in K_{-90-10a-k_1}$, we can choose spatial coordinates such that one coordinate axis is aligned with ω_a. The operator $R_1 P_{\omega_a}$ will then have L^1-mass of size $\sim 2^{-10a-k_1}$. We further distinguish between the following cases:

(2.1): At least one input at modulation $> 2^{k-1-20a-100}$. We can calculate

$$\begin{aligned}
&||P_0Q_j[P_{k_1,\omega_1}P_{k_1,\omega_2}\dots P_{k_1,\omega_a}P_{k_1}Q^+_{-k_1-20a-100<k_1-100}\psi_1\\
&\qquad\qquad P_{k_2,\tilde{\omega}_1}P_{k_2,\tilde{\omega}_2}\dots P_{k_2,\tilde{\omega}_a}R_1P_{k_2}Q^-_{<k_1-100}\psi_2]\\
&+P_0Q_j[P_{k_1,\omega_1}P_{k_1,\omega_2}\dots P_{k_1,\omega_a}R_1P_{k_1}Q^+_{-k_1-20a-100<k_1-100}\psi_1\\
&\qquad\qquad P_{k_2,\tilde{\omega}_1}P_{k_2,\tilde{\omega}_2}\dots P_{k_2\tilde{\omega}_a}P_{k_2}Q^-_{<k_1-100}\psi_2]||_{L^2_tL^2_x}\\
&\lesssim C^a2^{10a}2^{-10a}2^{\min\{\frac{j}{4},\frac{-10a}{4}\}}||P_{k_1,\omega_{a-1}}Q^+_{-k_1-20a<k_1-100}\psi_1||_{\dot{X}^{0,\frac{1}{2},\infty}_{k_1}}\\
&\qquad\qquad ||P_{k_2,\omega_{a-1}}Q^-_{<k_1-100}\psi_2||_{L^\infty_tL^2_x}.
\end{aligned}$$

From this one easily gets an exponential gain in $\min\{j,\,-a\}$, and one can then sum over ω_{a-1}, using Cauchy Schwarz, Plancherel's theorem and the definition of $S[k]$. One can sum over j, obtaining the claim of the lemma for fixed a, obtaining a small exponential gain in $-a$.

(2.2): Both inputs at modulation $< 2^{-k_1-20a-100}$. In this case, we use (14). Again we adapt the coordinates as above and compute

$$\begin{aligned}
&||P_0Q_j[P_{k_1,\omega_1}P_{k_1,\omega_2}\dots P_{k_1,\omega_a}P_{k_1}Q^+_{<-20a-k_1-100}\psi_1\\
&\qquad\qquad P_{k_2,\tilde{\omega}_1}P_{k_2,\tilde{\omega}_2}\dots P_{k_2,\tilde{\omega}_a}R_1P_{k_2}Q^-_{<k_1-100}\psi_2]\\
&+P_0Q_j[P_{k_1,\omega_1}P_{k_1,\omega_2}\dots P_{k_1,\omega_a}R_1P_{k_1}Q^+_{<-20a-k_1-100}\psi_1\\
&\qquad\qquad P_{k_2,\tilde{\omega}_1}P_{k_2,\tilde{\omega}_2}\dots P_{k_2\tilde{\omega}_a}P_{k_2}Q^-_{<k_1-100}\psi_2]||_{L^2_tL^2_x}\\
&\lesssim 2^{-10a}||P_{k_1,\omega_{a-1}}Q^+_{<-20a-k_1-100}\psi_1||_{S[k_1,\omega_{a-1}]}||P_{k_2,\tilde{\omega}_{a-1}}Q^-_{<k_1-100}\psi_2||_{S[k_2,-\tilde{\omega}_{a-1}]}.
\end{aligned}$$

One can sum over ω_{a-1} as before. Note that in this case, $\frac{j}{2} > -20a + O(1)$. The claim of the lemma follows again for fixed a with a small exponential gain in $-a$. In order to finish the proof, we note that it suffices to show that

$$\int_{\mathbf{R}^{2+1}}\sum_{\pm,\pm}\sum''_{\omega_M,\tilde{\omega}_M\in K_{-90-k_1-10M}}P_0[P_{\omega_1}P_{\omega_2}\dots P_{\omega_M}f^{\pm}P_{\tilde{\omega}_1}P_{\tilde{\omega}_2}\dots P_{\tilde{\omega}_M}g^{\pm}]\psi dxdt$$

converges to 0 as $M\to\infty$ for arbitrary $\psi\in\mathcal{S}(\mathbf{R}^{2+1})$. This follows easily by going to the Fourier side and observing that one obtains a double integral of a Schwartz function over an area decaying exponentially in M. ■

The next lemma is a less interesting technical tool:

Lemma 6.3. *Denote the operator $I = P_kQ_{<k+O(1)}$. Then for arbitrary Schwartz functions $\psi_1,\,\psi_2\in\mathcal{S}(\mathbf{R}^{2+1})$ and $\mu,\,\nu=0,1,2$, the following estimate holds:*

$$||P_0I[R_\nu P_{k_1}\psi_1R_\mu P_{k_2}\psi_2 - R_\mu P_{k_1}\psi_1R_\nu P_{k_2}\psi_2]||_{L^\infty_tL^2_x}\lesssim 2^{-\frac{|k_1-k_2|}{2}}\prod_{i=1,2}||P_{k_i}\psi_i||_{S[k_i]}$$

Proof : It follows easily from the 'Sobolev inequality' $||Q_j\psi||_{L_t^\infty L_x^2} \lesssim 2^{\frac{j}{2}}||Q_j\psi||_{L_t^2 L_x^2}$ and its dual. It suffices to prove the claim for $P_0[R_\nu P_{k_1}\psi_1 R_\mu P_{k_2}\psi_2]$, where not both $\mu, \nu = 0$. First assume $k_1 = k_2 + O(1) > O(1)$. Assume one input is at modulation $> 2^{k_1-100}$. Either it is hit by R_0, in which case the other input isn't, or else it isn't. In the former case, we compute

$$||P_0 I[R_0 P_{k_1} Q_{>k_1-100}\phi_1 P_{k_2} R_i\phi_2]||_{L_t^\infty L_x^2} \lesssim ||R_0 P_{k_1} Q_{>k_1-100}\phi_1||_{L_t^2 L_x^2}||P_{k_2} R_i\phi_2||_{L_t^\infty L_x^2}$$
$$\lesssim 2^{-\frac{k_1}{2}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.$$

In the latter case, if the other input is at large modulation, we are again in the former case. Otherwise, we compute

$$||P_0[R_0 P_{k_1} Q_{<k_1-100}\phi_1 P_{k_2} Q_{<k_2-100} R_i\phi_2]||_{L_t^\infty L_x^2} \lesssim ||P_{k_1}\psi_1||_{L_t^\infty L_x^2}||P_{k_2}\phi_2||_{L_t^\infty L_x^2}$$
$$\lesssim ||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.$$

The remaining frequency interactions are trivial variations of this. ∎

We next state another basic bilinear inequality, which is essentially identical to versions contained in [30], [18]:

Lemma 6.4. *Let ϕ_1, ϕ_2 be Schwartz functions. Then we have the inequality*

$$||P_k Q_j(P_{k_1}\phi_1 P_{k_2}\phi_2)||_{\dot{X}_k^{0,\frac{1}{2},\infty}} \lesssim 2^{\min\{k_1,k_2\}} 2^{\min\{\frac{j-\min\{k,k_1,k_2\}}{4+},0\}}$$
$$2^{\min\{\frac{\max\{k_1,k_2\}-j}{2},0\}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}$$

Also, if $k_2 >> k_1$, one has the estimate

$$||P_k(P_{k_1}\phi_1 P_{k_2} Q_{<a}\phi_2)||_{\dot{X}_k^{0,\frac{1}{2},1}} \lesssim 2^{k_1}(|\max\{\min\{k_2, a, \}, k_1\} - k_1| + 1)$$
$$||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}$$

Proof : The 2nd inequality is a straightforward consequence of the first. We give the proof again for **high-high interactions**, the other cases being mechanical repetitions of the same kind of argument and following as in [18], [30]: before beginning with the calculations, the following pivotal observation shall be crucial[24] (it is already implicit or explicit in [8], [33], [30], and our formulation is essentially that of [30]):

Geometric Observation:

Let ϕ, ψ be Schwartz functions. Consider the *microlocalized product*

$$P_{k_0} Q_{j_0}(P_{k_1} Q_{j_1}\phi P_{k_2} Q_{j_2}\psi)$$

[24]We will invoke it in the sequel without further mention.

If we assume that $\max_i\{j_i\} \leq \min\{k_i\} - C$ for some large C, the following conclusion applies: If we restrict the Fourier support of $P_{k_1}Q_{j_1}\psi_1$ further to the upper half-space $\tau > 0$ and an angular sector of opening $\kappa_1 \subset S^1$, where κ_1 has radius $2^{\frac{\min\{k_i\}-\max\{j_i\}}{2}-10}2^{k_0-\max\{k_1,k_2\}}$, then we can concurrently restrict the Fourier support of $P_{k_2}Q_{j_2}^{\pm}\psi$ to an angular sector of opening $\kappa_2 \subset S^1$ of the same radius and the following position relative to κ_1, without altering the output:

(1) When $\max\{j_i\} >> \max\{\{j_i\}\backslash \max\{j_i\}\}$, then we have

$$\mathrm{dist}(\kappa_1, \pm\kappa_2) \sim 2^{\frac{\min\{k_i\}-\max\{j_i\}}{2}}2^{k_0-\max\{k_1,k_2\}}$$

where the $\pm$ signs correspond to the sign in $P_{k_2}Q_{j_2}^{\pm}\psi$.

(2) When $\max\{j_i\} = \max\{\{j_i\}\backslash \max\{j_i\}\} + O(1)$, we have

$$\mathrm{dist}(\kappa_1, \pm\kappa_2) \lesssim 2^{\frac{\min\{k_i\}-\max\{j_i\}}{2}}2^{k_0-\max\{k_1,k_2\}}$$

with the same comment applying to $\pm$.

A similar conclusion applies when $P_{k_1}Q_{j_1}\phi$ is further restricted to the lower half-space $\tau < 0$. In other words, if for example we are in case 1 above, we have the equality

$$P_{k_0}Q_{j_0}(P_{k_1,\kappa_1}Q_{j_1}^{+}\phi P_{k_2}Q_{j_2}^{\pm}\psi) = \sum_{\kappa_2} P_{k_0}Q_{j_0}(P_{k_1,\kappa_1}Q_{j_1}^{+}\phi P_{k_2,\kappa_2}Q_{j_2}^{\pm}\psi)$$

where the sum is extended over all κ_2 satisfying the condition 1; there are only $O(1)$ many such caps.

Continuing with the proof of the lemma, we henceforth assume $k_1 = k_2 + O(1) \geq k + O(1)$. We first assume $j > k + 10$:

$$P_kQ_j(P_{k_1}\phi_1 P_{k_2}\phi_2) = P_kQ_j(P_{k_1}Q_{\geq j-10}\phi_1 P_{k_2}\phi_2)$$
$$+ P_kQ_j(P_{k_1}Q_{<j-10}\phi_1 P_{k_2}Q_{\geq j-10}\phi_2) + \sum_{\pm} P_kQ_j(P_{k_1}Q^{\pm}_{<j-10}\phi_1 P_{k_2}Q^{\pm}_{<j-10}\phi_2)$$

We estimate

$$||P_kQ_j(P_{k_1}Q_{\geq j-10}\phi_1 P_{k_2}\phi_2)||_{L_t^2L_x^2} \lesssim 2^k||P_{k_1}Q_{\geq j-10}\phi_1||_{L_t^2L_x^2}||P_{k_2}\phi_2||_{L_t^\infty L_x^2}$$
$$\lesssim 2^{k-\max\{\frac{j}{2}, j-\frac{k_1}{2}\}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}$$

The 2nd term on the right-hand side is estimated similarly. As to the third, it doesn't vanish only in case $j = k_1 + O(1)$:

$$
\begin{aligned}
&||\sum_{\pm} P_k Q_{k_1+O(1)}(P_{k_1} Q^{\pm}_{<j-10}\phi_1 P_{k_2} Q^{\pm}_{<j-10}\phi_2)||_{L_t^2 L_x^2} \\
&\leq ||\sum_{\pm} P_k Q_{k_1+O(1)}(P_{k_1} Q^{\pm}_{<2k-k_1}\phi_1 P_{k_2} Q^{\pm}_{<2k-k_1}\phi_2)||_{L_t^2 L_x^2} \\
&+ ||\sum_{\pm} P_k Q_{k_1+O(1)}(P_{k_1} Q^{\pm}_{2k-k_1\leq .<j-10}\phi_1 P_{k_2} Q^{\pm}_{<2k-k_1}\phi_2)||_{L_t^2 L_x^2} \\
&+ ||\sum_{\pm} P_k Q_{k_1+O(1)}(P_{k_1} Q^{\pm}_{<j-10}\phi_1 P_{k_2} Q^{\pm}_{2k-k_1\leq .<j-10}\phi_2)||_{L_t^2 L_x^2}
\end{aligned}
$$

The first of the preceding three terms is estimated as follows:

$$
\begin{aligned}
&||\sum_{\pm} P_k Q_{k_1+O(1)}(P_{k_1} Q^{\pm}_{<2k-k_1}\phi_1 P_{k_2} Q^{\pm}_{<2k-k_1}\phi_2)||_{L_t^2 L_x^2} \\
&= ||\sum_{\pm} \sum_{\kappa_{1,2}\in K_{k-k_1-10},\, \mathrm{dist}(\kappa_1,\kappa_2)\sim 2^{k-k_1}} \\
&\qquad ||P_k Q_{k_1+O(1)}(P_{k_1,\kappa_1} Q^{\pm}_{<2k-k_1}\phi_1 P_{k_2,\kappa_2} Q^{\pm}_{<2k-k_1}\phi_2)||_{L_t^2 L_x^2} \\
&\lesssim 2^{\frac{k}{2}} \sum_{\pm} (\sum_{\kappa_1\in K_{k-k_1-10}} ||P_{k_1,\kappa_1} Q^{\pm}_{<2k-k_1}\phi_1||^2_{S[k_1,\pm\kappa_1]})^{\frac{1}{2}} \\
&\qquad (\sum_{\kappa_2\in K_{k-k_2-10}} ||P_{k_2,\kappa_2} Q^{\pm}_{<2k-k_2}\phi_2||^2_{S[k_2,\pm\kappa_2]})^{\frac{1}{2}} \\
&\lesssim 2^{\frac{k}{2}} ||P_{k_1}\phi_1||_{S[k_1]} ||P_{k_2}\phi_2||_{S[k_2]}
\end{aligned}
$$

The 2nd and third term are estimated identically:

$$
\begin{aligned}
&||\sum_{\pm} P_k Q_{k_1+O(1)}(P_{k_1} Q^{\pm}_{2k-k_1\leq .<j-10}\phi_1 P_{k_2} Q^{\pm}_{<2k-k_1}\phi_2)||_{L_t^2 L_x^2} \\
&\lesssim 2^k \sum_{\pm} ||P_{k_1} Q^{\pm}_{2k-k_1\leq .<j-10}\phi_1||_{L_t^2 L_x^2} ||P_{k_2} Q^{\pm}_{<2k-k_1}\phi_2||_{L_t^\infty L_x^2} \\
&\lesssim 2^{\frac{k_1}{2}} ||P_{k_1}\phi_1||_{S[k_1]} ||P_{k_2}\phi_2||_{S[k_2]}
\end{aligned}
$$

We can thus restrict $j \leq k + 10$. We now use a further decomposition:

$$
\begin{aligned}
&P_k Q_j (P_{k_1}\phi_1 P_{k_2}\phi_2) = P_k Q_j (P_{k_1} Q_{\geq 2k-k_1}\phi_1 P_{k_2}\phi_2) \\
&+ P_k Q_j (P_{k_1} Q_{<2k-k_1}\phi_1 P_{k_2} Q_{\geq 2k-k_1}\phi_2) \\
&+ P_k Q_j (P_{k_1} Q_{<\min\{2k-k_1, j-C\}}\phi_1 P_{k_2} Q_{<\min\{j-C, 2k-k_1\}}\phi_2) \\
&+ P_k Q_j (P_{k_1} Q_{2k-k_1\geq .\geq j-C}\phi_1 P_{k_2} Q_{<2k-k_1}\phi_2) \\
&+ P_k Q_j (P_{k_1} Q_{<\min\{2k-k_1, j-C\}}\phi_1 P_{k_2} Q_{2k-k_1\geq .\geq j-C}\phi_2)
\end{aligned}
$$

The first two of the terms on the right-hand side are treated exactly as before. We estimate the third:

$$
\begin{aligned}
&||P_kQ_j(P_{k_1}Q_{<\min\{2k-k_1,j-C\}}\phi_1 P_{k_2}Q_{<\min\{j-C,2k-k_1\}}\phi_2)||_{L_t^2L_x^2}\\
&\leq \sum_{\pm,\pm}||\sum_{\kappa_{1,2}\in K_{\frac{j+k}{2}-k_1-10},\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j+k}{2}-k_1}}\\
&\qquad P_kQ_j(P_{k_1,\kappa_1}Q^{\pm}_{<\min\{2k-k_1,j-C\}}\phi_1 P_{k_2,\kappa_2}Q^{\mp}_{<\min\{j-C,2k-k_1\}}\phi_2)||_{L_t^2L_x^2}\\
&\lesssim 2^{k_1-\frac{j+k}{4}}\sum_{\pm,\pm}||\sum_{\kappa_{1,2}\in K_{\frac{j+k}{2}-k_1-10},\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j+k}{2}-k_1}}\\
&\qquad ||P_{k_1,\kappa_1}Q^{\pm}_{<\min\{2k-k_1,j-C\}}\phi_1||_{S[k_1,\pm\kappa_1]}||P_{k_2,\kappa_2}Q^{\pm}_{<\min\{2k-k_1,j-C\}}\psi_2||_{S[k_2,\pm\kappa_2]}
\end{aligned}
$$

Using the property

$$||P_{k,\kappa}Q^{\pm}_{<k}\psi||_{S[k,\pm\kappa]}\lesssim||P_k\psi||_{\dot{X}_k^{0,\frac{1}{2},1}}$$

as well as Plancherel's theorem, one easily verifies that

$$\Big(\sum_{\kappa\in K_{\frac{j+k}{2}-k_1-10}}||P_{k_1,\kappa}Q^{\pm}_{<\min\{2k-k_1,j-C\}}\psi||^2_{S[k_1,\pm\kappa]}\Big)^{\frac{1}{2}}\lesssim|j-k|||P_{k_1}\psi||_{S[k_1]}$$

Thus one obtains the estimate

$$
\begin{aligned}
&||P_kQ_j(P_{k_1}Q_{<\min\{2k-k_1,j-C\}}\phi_1 P_{k_2}Q_{<\min\{j-C,2k-k_1\}}\phi_2)||_{\dot{X}_k^{0,\frac{1}{2},\infty}}\\
&\lesssim|j-k|^2 2^{k_1}2^{\frac{j-k}{4}}\prod_{i=1,2}||P_{k_i}\psi_i||_{S[k_i]}
\end{aligned}
$$

Next, one estimates using the improved Bernstein's inequality

$$
\begin{aligned}
&||P_kQ_j(P_{k_1}Q_{2k-k_1\geq.\geq j-C}\phi_1 P_{k_2}Q_{<2k-k_1}\phi_2)||_{L_t^2L_x^2}\\
&\lesssim 2^k2^{\frac{j-k}{4}}||P_{k_1}Q_{2k-k_1\geq.\geq j-C}\phi_1||_{L_t^2L_x^2}||P_{k_2}Q_{<2k-k_1}\phi_2||_{L_t^\infty L_x^2}\\
&\lesssim 2^{k-\frac{k+j}{4}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.
\end{aligned}
$$

■

We need one more important bilinear estimate, which is the analogue in our context of the embedding $X^{s,\theta}\times X^{s-1,\theta-1}\subset X^{s-1,\theta-1}$ in [9] valid in the context of $\mathbf{R}^{n+1}$, $n\geq 2$ for $s>\frac{n}{2}$, $\theta>\frac{1}{2}$:

Lemma 6.5. *Let $r\leq 0$, and let F, ψ be Schwartz functions at frequencies k_1,k_2 respectively where $k_1=k_2+O(1)$. Let $k\leq k_1+O(1)$, $j\leq r+k$, $l=r+k$. Then we have the inequality*

$$||P_kQ_{<l}[Q_jFQ_{<2k+r-k_1}\psi]||_{N[k]}\lesssim 2^{\delta r}||F||_{\dot{X}_{k_1}^{0,-\frac{1}{2},\infty}}||\psi||_{S[k_2]}$$

Moreover, we also have the more crude version (no restrictions on j)

$$||P_k[Q_jF\psi]||_{N[k]} \lesssim ||F||_{\dot{X}_{k_1}^{0,-\frac{1}{2},\infty}}||\psi||_{\dot{X}_{k_2}^{0,\frac{1}{2},1}}$$

Next, assume $j \leq \min\{k_1,k_2\}+O(1)$. Then for suitable $\delta > 0$ we have the inequality

$$||P_{\max\{k_{1,2}\}+O(1)}[P_{k_1}\psi P_{k_2}Q_jF]||_{N[k_2]} \lesssim 2^{\delta(j-\min\{k_1,k_2\})}2^{k_1}||P_{k_1}\psi_1||_{S[k_1]}||P_{k_2}F||_{\dot{X}^{0,-\frac{1}{2},\infty}}$$

Proof : We prove the first inequality, corresponding to **high-high interactions**. The last inequality is contained in [30]. The restriction on the modulation of the output will allow us to apply a refined version of Bernstein's inequality. The restriction on the modulation of ψ prevents logarithmic factors of the form $|k-k_1|$. This is important since we don't obtain an exponential gain in the difference $k-k_1$ as in other versions of the lemma valid in dimensions 3 and higher (see [18]).
We rescale to $k_1 = k_2 + O(1) = 0$, whence $k \leq O(1)$.

(1): Estimating $P_kQ_{<l}(Q_jFQ_{2k+r-k_1>.\geq j-C}\psi)$. Note that our assumptions about the modulations of output and the inputs imply that provided we restrict the microsupport of F to be contained in an angular sector given by a cap κ_1 of size $2^{\frac{l+k}{2}-10}$, and provided we also microlocalize F to either the upper or lower half-space $\tau >< 0$, we can concurrently restrict the microsupport of ψ to a cap κ_2 of similar size and either the lower or upper half-space such that $\pm\kappa_1, \pm\kappa_2$ are at distance $\leq C2^{\frac{l+k}{2}}$, the signs being assigned according to whether the corresponding input is microlocalized to the upper or lower half-space. Therefore we can estimate this term as follows:

$$\begin{aligned}
&||P_kQ_{<l}(Q_jFQ_{2k+r-k_1>.\geq j-C}\psi)||_{N[k]} \\
&\leq \sum_{\pm,\pm,\pm}\ \sum_{\kappa_{1,2}\in K_{\frac{l+k}{2}-10},\,\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\lesssim 2^{\frac{l+k}{2}}}\ \sum_{\kappa\in K_{\frac{l-k}{2}-10},\,\mathrm{dist}(\pm\kappa,\pm\kappa_1)\lesssim 2^{\frac{l-k}{2}}} \\
&\qquad 2^{-k}||P_{k,\kappa}Q^{\pm}_{<l}(P_{k_1,\kappa_1}Q^{\pm}_jFQ_{2k+r-k_1>.\geq j-C}P_{k_2,\kappa_2}\psi)||_{L^1_tL^2_x} \\
&\lesssim 2^{\frac{l-k}{4}}\sum_{\pm,\pm,\pm}\ \sum_{\kappa_{1,2}\in K_{\frac{l+k}{2}-10},\,\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)}||P_{k_1,\kappa_1}Q^{\pm}_jF||_{L^2_tL^2_x} \\
&\qquad ||P_{k_2,\kappa_2}Q^{\pm}_{2k+r-k_1>.\geq j-C}\psi||_{L^2_tL^2_x} \\
&\lesssim 2^{\frac{l-k}{4}}||P_{k_1}Q_jF||_{\dot{X}_{k_1}^{0,-\frac{1}{2},\infty}}||P_{k_2}Q_{2k+r-k_1>.\geq j-C}\psi||_{\dot{X}_{k_2}^{0,\frac{1}{2},\infty}} \\
&\lesssim 2^{\frac{r}{4}}||F||_{\dot{X}_{k_1}^{0,-\frac{1}{2},\infty}}||\psi||_{S[k_2]}
\end{aligned}$$

(2): $P_kQ_{l\geq.\geq j-C}(FQ_{<\min\{j-C,2k+r-k_1\}}\psi)$. This is similar to the preceding case.

(3): The estimate for $P_kQ_{<j-C}(Q_jFQ_{<\min\{j-C,2k+r-k_1\}}\psi)$. Arguing as in **(1)**, we can microlocalize F, $Q_{<\min\{j-C,2k+r-k_1\}}\psi$ to either half-space $\pm\tau > 0$ and caps

$\kappa_{1,2}\in K_{\frac{j+k}{2}-10}$ such that $\text{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j+k}{2}}$.

$$
\begin{aligned}
&||P_kQ_{<j-C}(Q_jFQ_{<\min\{j-C,2k+r-k_1\}}\psi)||_{N[k]}\\
&\leq\sum_{\pm,\pm}\sum_{\kappa_{1,2}\in K_{\frac{j+k}{2}-10},\,\text{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j+k}{2}}}\\
&\qquad\qquad ||P_kQ_{<j-C}(P_{k_1,\kappa_1}Q_j^{\pm}FP_{k_2,\kappa_2}Q^{\pm}_{<\min\{j-C,2k+r-k_1\}}\psi)||_{N[k]}
\end{aligned}
$$

For fixed $\kappa_{1,2}$, we can further decompose this as follows, and use (20):

$$
\begin{aligned}
&||P_kQ_{<j-C}(P_{k_1,\kappa_1}Q_j^{\pm}FP_{k_2,\kappa_2}Q^{\pm}_{<\min\{j-C,2k+r-k_1\}}\psi)||_{N[k]}\\
&\leq 2^{-k}\sum_{\pm}\sum_{\kappa\in K_{\frac{j-k}{2}-10},\,\text{dist}(\pm\kappa,\pm\kappa_1)\sim 2^{\frac{j-k}{2}}}||P_{k,\kappa}Q^{\pm}_{<j-C}(P_{k_1,\kappa_1}Q_j^{\pm}F\\
&\qquad\qquad P_{k_2,\kappa_2}Q^{\pm}_{<\min\{j-C,2k+r-k_1\}}\psi)||_{NFA[\kappa]}\\
&\lesssim 2^{-k}2^{-\frac{j-k}{2}}2^{\frac{j+k}{4}}||P_{k,\kappa_1}Q^{\pm}_{<j-C}P_{k_1,\kappa_1}Q_j^{\pm}F||_{L_t^2L_x^2}\\
&\qquad\qquad ||P_{k_2,\kappa_2}Q^{\pm}_{<\min\{j-C,2k+r-k_1\}}\psi||_{S[k_2,\kappa_2]}
\end{aligned}
$$

where we have exploited the finiteness of admissible caps κ for fixed $\kappa_{1,2}$. Now we use the Cauchy-Schwarz inequality as well as Plancherel's theorem and the definition of $S[k_2]$. The only issue we have to be careful about is the adjustment of the modulation cut-off for the 2nd input: we decompose $P_{k_2,\kappa_2}Q^{\pm}_{<\min\{j-C,2k+r-k_1\}}\psi_2$ into a part very close to the light cone (modulation $<2^{k+j-k_2}$) and an error term covering $\leq|j-k|+O(1)$ many dyadic modulation intervals. The first contribution is immediate from the definition of $S[k]$. As to the 2nd, we note that

$$
\begin{aligned}
&\sum_{\pm,\pm}\sum_{\kappa_{1,2}\in K_{\frac{j+k}{2}-10},\,\text{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j+k}{2}}}\\
&||P_{k_1,\kappa_1}Q_j^{\pm}F||_{L_t^2L_x^2}||P_{k_2,\kappa_2}Q^{\pm}_{j+k-k_2<.<2k+r-k_2}\psi||_{S[k_2,\pm\kappa_2]}\\
&\leq\sum_{\pm,\pm}\sum_{\kappa_{1,2}\in K_{\frac{j+k}{2}-10},\,\text{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j+k}{2}}}\sum_{j+k-k_2<a<2k+r-k_2}\\
&||P_{k_1,\kappa_1}Q_j^{\pm}F||_{L_t^2L_x^2}||P_{k_2,\kappa_2}Q_a^{\pm}\psi||_{S[k_2,\pm\kappa_2]}\\
&\lesssim\sum_{j+k-k_2<a<2k+r-k_2}||P_{k_1}Q^{\pm}F||_{L_t^2L_x^2}||P_{k_2}\psi||_{\dot{X}_{k_2}^{0,\frac12,\infty}}\\
&\lesssim|j-k|||F||_{L_t^2L_x^2}||P_{k_2}\psi||_{S[k_2]}
\end{aligned}
$$

This immediately implies that

$$
\begin{aligned}
&||P_k Q_{<j-C}(Q_j F Q_{<\min\{j-C,2k+r-k_1\}}\psi)||_{N[k]}\\
&\lesssim |j-k| 2^{\frac{j-k}{4}} ||F||_{\dot{X}_{k_1}^{0,-\frac{1}{2},\infty}} ||\psi||_{S[k_2]}\\
&\lesssim 2^{\frac{r}{4+}} ||F||_{\dot{X}_{k_1}^{0,-\frac{1}{2},\infty}} ||\psi||_{S[k_2]}
\end{aligned}
$$

■

We close this section with a lemma extending lemma 3.1 to **improved type Strichartz norms** as introduced by Klainerman-Tataru in [16]. We first need the following simple

Lemma 6.6. *(***Improved Bernstein a la Klainerman-Tataru***): Let ψ be a Schwartz function. Then provided $j < k + O(1)$, and we let $C_{k,l}$, $l < -10$, denote a finitely overlapping cover of the region $\{\xi||\xi| \sim 2^k\}$ with discs c of radius 2^{k+l}, one has*

$$(\sum_{c\in C_{k,l}} ||P_c Q_j \psi||_{L_t^2 L_x^\infty}^2)^{\frac{1}{2}} \lesssim 2^{\frac{l}{2}} 2^k 2^{\frac{j-k}{4}} ||P_k\psi||_{L_t^2 L_x^2}$$

where P_c microlocalizes to the disc c.

Proof : We replicate the argument in [30] with one extra wrinkle: we may put $j = 0$, whence $k >> 1$. Construct a Schwartz function $a(t)$ whose Fourier transform is supported in $\tau << 1$, as well as satisfying

$$1 = \sum_{s\in\mathbf{Z}} a^3(t-s)$$

$\forall t \in \mathbf{R}$. Then we have

$$
\begin{aligned}
&||P_c Q_0\psi||_{L_t^2 L_x^\infty} \le ||\sum_s a^3(t-s) P_c Q_0 \psi||_{L_t^2 L_x^\infty}\\
&\le (\sum_s ||a^2(t-s)P_c Q_0\psi||_{L_t^2 L_x^\infty}^2)^{\frac{1}{2}} \lesssim (\sum_s ||a(t-s)P_c Q_0\psi||_{L_t^4 L_x^\infty}^2)^{\frac{1}{2}}
\end{aligned}
$$

Now one notes that the function $a(t-s)P_c Q_0\psi$ satisfies almost the same assumptions about modulation (~ 1) and frequency localization as $P_c Q_0\psi$. Therefore, we can apply the standard improved Strichartz type inequality by Klainerman-Tataru [16] to estimate

$$||a(t-s)P_c Q_0\psi||_{L_t^4 L_x^\infty} \lesssim 2^{\frac{3k}{4}} 2^{\frac{l}{2}} ||P_c\psi||_{\dot{X}_k^{0,\frac{1}{2},\infty}}$$

Thus

$$||P_cQ_0\psi||_{L_t^2L_x^\infty} \leq C2^{\frac{3k}{4}}2^{\frac{l}{2}}(\sum_s ||a(t-s)Q_0P_c\psi||^2_{\dot{X}_k^{0,\frac{1}{2},\infty}})^{\frac{1}{2}}$$
$$\lesssim 2^{\frac{3k}{4}}2^{\frac{l}{2}}||P_cQ_0\psi||_{L_t^2L_x^2}$$

One now obtains the statement of the lemma via Plancherel's theorem. ■

The next lemma deals with the control of certain frequency localized Strichartz and improved-Strichartz type norms in terms of the spaces $S[k]$:

Lemma 6.7. *Let ψ be a Schwartz function. Further, for every $l < -10$, let $C_{k,l}$ be a (uniformly) finitely overlapping cover of the region $|\xi| \sim 2^k$ in Fourier space by discs of radius 2^{k+l}. Denote by P_c the Fourier multiplier microlocalizing to such a disc $c \in C_{k,l}$. Then for $8 \geq p > 4$ we have the inequalities*

$$(\sum_{c\in C_{0,l}} ||P_c\psi||^2_{L_t^pL_x^\infty})^{\frac{1}{2}} \leq C_p 2^{l(\frac{3}{4+}-\frac{2}{p})}||\psi||_{S[0]}$$

In particular

$$(\sum_{c\in C_{0,l}} ||P_c\psi||^2_{L_t^8L_x^\infty})^{\frac{1}{2}} \lesssim 2^{\frac{l}{2+}}||\psi||_{S[0]}$$

Rescaling this, one obtains:

$$(\sum_{c\in C_{k,l}} ||P_c\psi||^2_{L_t^8L_x^\infty})^{\frac{1}{2}} \lesssim 2^{\frac{l}{2+}}2^{\frac{7k}{8}}||\psi||_{S[k]}$$

Therefore, interpolating with $L_t^\infty L_x^2$, one obtains

$$||P_0\psi||_{L_t^pL_x^q} \lesssim ||\psi||_{S[0]}$$

provided $\frac{1}{p}+\frac{1}{2q} < \frac{1}{4}$, $p > 4$.

Proof : Let $p \geq 2$. By the triangle inequality

$$(\sum_{c\in C_{0,l}} ||P_c\psi||^2_{L_t^pL_x^\infty})^{\frac{1}{2}} \leq (\sum_{c\in C_{0,l}} ||P_cQ_{\geq -10}\psi||^2_{L_t^pL_x^\infty})^{\frac{1}{2}} +$$
$$\sum_{\pm}(\sum_{c\in C_{0,l}} ||P_cQ^{\pm}_{<-10}\psi||^2_{L_t^pL_x^\infty})^{\frac{1}{2}}$$

Next, by another application of the triangle inequality

$$(\sum_{c\in C_{0,l}} ||P_cQ_{\geq -10}\psi||^2_{L_t^pL_x^\infty})^{\frac{1}{2}} \leq \sum_{j\geq -10}(\sum_{c\in C_{0,l}} ||P_cQ_j\psi||^2_{L_t^pL_x^\infty})^{\frac{1}{2}}$$

Now we can estimate

$$\begin{aligned}&||P_cQ_j\psi||^2_{L_t^pL_x^\infty} \leq ||P_cQ_j\psi P_cQ_j\psi||_{L_t^{\frac{p}{2}}L_x^\infty}\\&\leq ||P_{\tilde{c}}Q_{\leq j+O(1)}[P_cQ_j\psi P_cQ_j\psi]||_{L_t^{\frac{p}{2}}L_x^\infty}\end{aligned}$$

where $\tilde{c}=\tilde{c}(c)$ is a disc of approximately the same size as c at frequency ~ 1. In particular, we have

$$\begin{aligned}&||P_cQ_j\psi||^2_{L_t^4L_x^\infty} \leq C2^{2l}||P_cQ_j\psi P_cQ_j\psi||_{L_t^2L_x^1}\\&\leq C2^{2l-j}||P_cQ_j\psi||_{\dot{X}^{-\frac{1}{2},1,2}}||P_c\psi||_{L_t^\infty L_x^2}\end{aligned}$$

Now one can sum over $c\in C_{0,l}$, using Cauchy-Schwarz as well as Plancherel's theorem and the definition of $S[k,\kappa]$, and finally one sums over $j\geq -10$, yielding (more than) the required result for $p=4$. One proceeds similarly for $p=8$, and the full result follows by interpolation.
Hence we can move on to the more interesting case when the input lives close to the light cone. We decompose

$$\begin{aligned}&\sum_{\pm}||P_cQ^{\pm}_{<-10}\psi||^2_{L_t^pL_x^\infty} \leq \sum_{\pm}\sum_{j<O(1)}||P_{\tilde{c}}Q_j^{\pm}[P_cQ^{\pm}_{<j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L_t^{\frac{p}{2}}L_x^\infty}\\&+\sum_{\pm}\sum_{j<O(1)}||P_{\tilde{c}}Q_j^{\pm}[P_cQ^{\pm}_{-10>.\geq j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L_t^{\frac{p}{2}}L_x^\infty}\\&+\sum_{\pm}\sum_{j<O(1)}||P_{\tilde{c}}Q_j^{\pm}[P_cQ^{\pm}_{<-10}\psi P_cQ^{\pm}_{-10>.\geq j-C}\psi]||_{L_t^{\frac{p}{2}}L_x^\infty}\end{aligned}\tag{24}$$

Consider the first term: note that we have

$$\begin{aligned}&\sum_{\pm}P_{\tilde{c}}Q_j^{\pm}[P_cQ^{\pm}_{<j-C}\psi P_cQ^{\pm}_{<j-C}\psi]\\&=\sum_{\pm}\sum_{\kappa_{1,2}\in K_{\frac{j}{2}-10},\,\mathrm{dist}(\kappa_1,\kappa_2)\sim 2^{\frac{j}{2}}}P_{\tilde{c}}Q_j^{\pm}[P_cP_{0,\kappa_1}Q^{\pm}_{<j-C}\psi P_cP_{0,\kappa_2}Q^{\pm}_{<j-C}\psi]\end{aligned}$$

This immediately implies that $j<2l+O(1)$, for otherwise the above vanishes(of course this simply expresses the curvature of the cone). Now we use the improved Bernstein's inequality to estimate

$$\begin{aligned}&\sum_{\pm}||P_{\tilde{c}}Q_j^{\pm}[P_cQ^{\pm}_{<j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L_t^2L_x^\infty}\\&\leq C2^{\frac{j}{4}}2^{\frac{l}{2}}\sum_{\pm}||P_{\tilde{c}}Q_j^{\pm}[P_cQ^{\pm}_{<j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L_t^2L_x^2}\end{aligned}$$

$$\leq C2^{\frac{j}{4}}2^{\frac{l}{2}}\sum_{\pm}\sum_{\kappa_{1,2}\in K_{\frac{j}{2}-10},\,\mathrm{dist}(\kappa_1,\kappa_2)\sim 2^{\frac{j}{2}}}||P_cP_{0,\kappa_1}Q^{\pm}_{<j-C}\psi P_cP_{0,\kappa_2}Q^{\pm}_{<j-C}\psi||_{L^2_tL^2_x}$$
$$\leq \sum_{\pm}C2^{\frac{l}{2}}(\sum_{\kappa_1\in K_{\frac{j}{2}-10}}||P_cP_{0,\pm\kappa_1}Q^{\pm}_{<j-C}\psi||^2_{S[0,\kappa_1]})$$

Next, we do the exact same thing for $L^8_tL^\infty_x$ except that now we have to estimate $||(P_c\psi)^2)||_{L^4_tL^\infty_x}$. Therefore, we employ the improved Strichartz inequality of Klainerman-Tataru instead of the improved Bernstein's inequality. One obtains:

$$\sum_{\pm}||P_{\tilde{c}}Q^{\pm}_j[P_cQ^{\pm}_{<j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L^4_tL^\infty_x}$$
$$\leq C2^{\frac{l}{2}}2^{\frac{j}{4}}\sum_{\pm}(\sum_{\kappa_1\in K_{\frac{j}{2}-10}}||P_cP_{0,\pm\kappa_1}Q^{\pm}_{<j-C}\psi||^2_{S[0,\kappa_1]})$$

One can interpolate between the two preceding inequalities to obtain the statement

$$\sum_{\pm}||P_{\tilde{c}}Q^{\pm}_j[P_cQ^{\pm}_{<j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L^p_tL^\infty_x}$$
$$\leq C2^{\frac{l}{2}}2^{j(\frac{1}{2}-\frac{1}{p})}\sum_{\pm}(\sum_{\kappa_1\in K_{\frac{j}{2}-10}}||P_cP_{0,\pm\kappa_1}Q^{\pm}_{<j-C}\psi||^2_{S[0,\kappa_1]})$$

where $2\leq p\leq 4$. Next, one can sum over $c\in C_{0,l}$ and apply the definition of the $S[k]$ to conclude

$$\sum_{\pm}\sum_{c\in C_{0,l}}||P_{\tilde{c}}Q^{\pm}_j[P_cQ^{\pm}_{<j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L^p_tL^\infty_x}$$
$$\leq C2^{\frac{l}{2+}}2^{j(\frac{1}{2}-\frac{1}{p})}||P_0\psi||^2_{S[0]}$$

Recalling that $j<2l+O(1)$, one can now sum over j to obtain the result

$$\sum_{\pm}\sum_{j<2l+O(1)}\sum_{c\in C_{0,l}}||P_{\tilde{c}}Q^{\pm}_j[P_cQ^{\pm}_{<j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L^p_tL^\infty_x}\leq C2^{\frac{l}{2+}}2^{l(1-\frac{2}{p})}||P_0\psi||^2_{S[0]}$$

provided $4\geq p>2$.
We move on to the other terms in (24). We estimate

$$||P_{\tilde{c}}Q^{\pm}_j[P_cQ^{\pm}_{-10>.\geq j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L^2_tL^\infty_x}$$
$$\leq C2^{\frac{l}{2}}2^{\frac{j}{4}}||P_{\tilde{c}}Q^{\pm}_j[P_cQ^{\pm}_{-10>.\geq j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L^2_tL^2_x}$$
$$\leq \sum_{-10>a\geq j-C}C2^{\frac{j}{4}}2^{\frac{l}{2}}||P_cQ^{\pm}_a\psi||_{L^2_tL^\infty_x}||P_cQ^{\pm}_{<j-C}\psi||_{L^\infty_tL^2_x}$$
$$\leq \sum_{a>j-C}C2^l2^{\frac{j-a}{4}}||P_cQ^{\pm}_a\psi||_{\dot{X}_0^{0,\frac{1}{2},\infty}}||P_c\psi||_{L^\infty_tL^2_x}$$

We have used the improved Bernstein's inequality a la Klainerman-Tataru twice. Similarly, again using the improved Bernstein's inequality for the large-modulation input but the improved Strichartz inequality for the output, we conclude that

$$\begin{aligned}
&||P_{\tilde{c}}Q_j^{\pm}[P_cQ^{\pm}_{-10>.\geq j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L_t^4L_x^\infty}\\
&\leq C2^{\frac{l}{2}}2^{\frac{j}{2}}||P_cQ^{\pm}-10>.\geq j-C\psi P_cQ^{\pm}_{<j-C}\psi||_{L_t^2L_x^2}\\
&\leq C\sum_{a\geq j-C}2^{\frac{l}{2}}2^{\frac{j}{2}}||P_cQ_a^{\pm}\psi||_{L_t^2L_x^\infty}||P_c\psi||_{L_t^\infty L_x^2}\\
&\leq C\sum_{a\geq j-C}2^l2^{\frac{j}{2}-\frac{a}{4}}||P_c\psi||_{\dot{X}_0^{0,\frac{1}{2},\infty}}||P_c\psi||_{L_t^\infty L_x^2}
\end{aligned}$$

Interpolating this with the previous estimate implies

$$\begin{aligned}
&||P_{\tilde{c}}Q_j^{\pm}[P_cQ^{\pm}_{-10>.\geq j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||_{L_t^pL_x^\infty}\\
&\leq C2^l\sum_{a\geq j-C}2^{j(\frac{3}{4}-\frac{1}{p})-\frac{a}{4}}||P_cQ_a^{\pm}\psi||_{\dot{X}_0^{0,\frac{1}{2},\infty}}||P_c\psi||_{L_t^\infty L_x^2}
\end{aligned}$$

Now one sums over c, using Cauchy-Schwartz and Plancherel as well as the definition of the $S[k,\kappa]$, then executes the summation over a and subsequently sums over $j<10$. There results

$$\sum_{c\in C_{0,l}}\sum_{j<10}||P_{\tilde{c}}Q_j^{\pm}[P_cQ^{\pm}_{-10>.\geq j-C}\psi P_cQ^{\pm}_{<j-C}\psi]||^2_{L_t^pL_x^\infty}\leq C2^l||P_0\psi||^2_{S[0]}$$

finally, the last term in (24) is dealt with similarly, hence left out. This establishes the claims of the lemma. ■

6.8. **Trilinear estimates.** Before beginning with the estimation of our trilinear null-forms, we state here a deep trilinear null-form inequality due to T. Tao (it constitutes the analytic core of [30]), which we formulate to meet our needs:

Theorem 6.9. *Let $\psi_{1,2,3}\in\mathcal{S}(\mathbf{R}^{2+1})$. Then, provided we have*

$$O(1)\geq\max\{k_1,k_2,k_3\}\geq k_2\geq\min\{k_1,k_3\}+O(1),$$

the following inequality holds[25]*:*

$$\begin{aligned}
&||P_0[P_{k_1}R_\nu\psi_1P_{k_2}\psi_2R^\nu P_{k_3}\psi_3]||_{N[0]}\\
&\qquad\qquad\lesssim 2^{\delta(\min\{k_1,k_3\}-\max\{\min\{k_1,k_2\},\min\{k_2,k_3\}\})}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}
\end{aligned}$$

Also, provided $k_1\leq k_2=k_3+O(1)\geq O(1)$, then

$$||P_0[P_{k_1}R_\nu\psi_1P_{k_2}\nabla^{-1}\psi_2R^\nu P_{k_3}\psi_3]||_{N[0]}\lesssim 2^{-\delta k_2}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}$$

[25] The first inequality is significantly more difficult to prove than the 2nd.

The difficult proof is contained in [30]. Our null-forms have a different schematic form, namely $\nabla_{x,t}[\psi\nabla^{-1}[\psi^2]]$. In particular, we lose one degree of smoothness for high frequencies. This forces us to pay special attention to destructive resonance phenomena inside the expression. The most difficult case occurs when there is destructive resonance inside the inner square bracket, on account of the operator ∇^{-1}. Indeed, not even the apparent $Q_{\nu j}$-structure seems enough to counteract this, and we have to treat the expression as a genuine trilinear null-form. We shall have to take advantage of an apparently new nontrivial cancellation in the proof of theorem 6.11, as well as our modification of the spaces $S[k]$. Our estimates are just enough to recover the frequency envelope and complete the bootstrapping. We do not necessarily obtain exponential gains in in the largest frequencies as in theorem 6.9, but only in the intermediate frequencies. As discussed in section 5.2, writing the trilinear null-form schematically as $\nabla_{x,t}[\psi\nabla^{-1}[\psi^2]]$, we distinguish between the case when the inner bracket $[,]$ is microlocalized far away from the light cone (i. e. apply $(1-I)$) and the opposite. We treat the easier former case first. Interestingly, even this 'elliptic case' offers some difficulties, as the time derivative on the outside may cause losses (frequency localization doesn't help), and we again have to treat this as a genuine trilinear null-form. We have the following:

Theorem 6.10. *Let $\psi_{1,2,3}\in\mathbf{R}^{2+1}$. Then, for integers $k_{1,2,3}$ and suitable $\delta_{1,2}>0$, the following inequality holds:*

$$
\begin{aligned}
&||\nabla_{x,t}P_0[P_{k_1}\psi_1\nabla^{-1}(1-I)[R_\nu P_{k_2}\psi_2R_j\psi_3-R_jP_{k_2}\psi_2R_\nu\psi_3]]||_{N[0]}\\
&\lesssim 2^{\delta_1\min\{-\min\{k_1,k_2,k_3\},0\}}\prod_i 2^{\delta_2\min\{\max_{j\neq i}\{k_i,k_i-k_j\},0\}}\prod_l ||P_{k_l}\psi_l||_{S[k_l]}
\end{aligned}
$$

Remark: One checks that this estimate implies the following: Assume that $||P_k\psi_i||\leq c_k$, $k\in\mathbf{Z}$, where $\{c_k\}_{k\in\mathbf{Z}}$ is a 'sufficiently flat' frequency envelope. Then

$$
\begin{aligned}
&\sum_{k_{1,2,3}\in\mathbf{Z}}||\nabla_{x,t}P_0[P_{k_1}\psi_1\nabla^{-1}(1-I)[R_\nu P_{k_2}\psi_2R_j\psi_3-R_jP_{k_2}\psi_2R_\nu\psi_3]]||_{N[0]}\\
&\lesssim c_0(\sum_{k\in\mathbf{Z}}c_k^2)
\end{aligned}
$$

Proof : We distinguish between the cases $k_1\geq 10$, $k_1\in[-10,10]$, $k_1<-10$.

(1): $k_1\geq 10$. In this case, we can rewrite the expression schematically[26] as

$$
\sum_{l\geq k_1+5}\nabla_{x,t}P_0[P_{k_1}\psi_1P_{[k_1-5,k_1+5]}\nabla^{-1}Q_l(1-I)[,]]
$$

We now distinguish further between the following cases:

[26]The inner square bracket $[,]$ stands for $[R_\nu P_{k_2}\psi_2R_jP_{k_3}\psi_3-R_jP_{k_2}\psi_2R_\nu P_{k_3}\psi_3]$.

(1.1): *Output at modulation* < 1: This immediately implies that the first input $P_{k_1}\psi_1$ is at modulation $\geq 2^{l-10}$. Using lemma 6.2 as well as Bernstein's inequality (we shall do so without further mention in the future), we estimate

$$\begin{aligned}
&||\nabla_{x,t}P_0Q_{<0}[P_{k_1}Q_{\geq l-10}\psi_1 P_{[k_1-5,k_1+5]}\nabla^{-1}Q_l(1-I)[,]]||_{N[0]}\\
&\lesssim ||\nabla_{x,t}P_0Q_{<0}[P_{k_1}Q_{\geq l-10}\psi_1 P_{[k_1-5,k_1+5]}\nabla^{-1}Q_l(1-I)[,]]||_{L_t^1L_x^2}\\
&\lesssim ||P_{k_1}Q_{\geq l-10}\psi_1||_{L_t^2L_x^2}||\nabla^{-1}Q_l[,]||_{L_t^2L_x^2}\\
&\lesssim |l|^2 2^{-l}2^{-\frac{|k_2-k_3|}{2}}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]},
\end{aligned}$$

which upon summing over $l \geq k_1+10$ yields an estimate of the desired form.

(1.2): *Output at modulation* ≥ 1. This is somewhat more complicated since the operator $\nabla_{x,t}$ may now cause a loss. We place this component of the expression into the space $\dot{X}_0^{-\frac{1}{2},-1,2}\cap\partial_t(L_t^M\dot{H}^{-(1-\frac{1}{M})})$. First, letting N be a large number and $\frac{1}{1+}+\frac{1}{N}=1$, and additionally assuming both inputs of the inner square bracket to be 'hyperbolic' in the sense that their modulation is smaller than their frequency, we can use lemma 3.1 to compute:

$$\begin{aligned}
&||P_0Q_{\geq 0}[P_{k_1}\psi_1\nabla^{-1}(1-I)[,]]||_{L_t^ML_x^2}\\
&\lesssim ||P_{k_1}\psi_1||_{L_t^ML_x^N}||\nabla^{-1}(1-I)P_{k_1+O(1)}[,]||_{L_t^\infty L_x^{1+}}\\
&\lesssim 2^{-\delta_1k_1}2^{-\delta_2|k_2-k_3|}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]}
\end{aligned}$$

for suitable $\delta_{1,2}>0$. If R_0 hits an 'elliptic' input (i. e. its modulation outweighs its frequency) inside the inner square bracket, one places this into $L_t^ML_x^2$, and the first input into $L_t^\infty L_x^2$. The simple details are omitted. Next, freezing the inner bracket to modulation $\sim 2^l$, $l\geq k_1+10$, we distinguish between the following subcases:
(1.2.a): *Output at modulation* $\leq 2^{l+10}$. Observe that either the whole output is at modulation $\geq 2^{l-10}$, or else the first input $P_{k_1}\psi_1$ is at modulation $\geq 2^{l-10}$. In the latter case, we estimate

$$\begin{aligned}
&||\nabla_{x,t}P_0Q_{[0,l-10]}[P_{k_1}Q_{\geq l-10}\psi_1\nabla^{-1}Q_l(1-I)[,]]||_{\dot{X}_0^{-\frac{1}{2},-1,2}}\\
&\lesssim 2^{\frac{l}{2}}||P_0[P_{k_1}Q_{\geq l-10}\psi_1\nabla^{-1}Q_l(1-I)[,]]||_{L_t^1L_x^2}\\
&\lesssim 2^{-\frac{l}{2+}}2^{-\frac{|k_2-k_3|}{2}}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]}
\end{aligned}$$

This can again be summed over $l\geq k_1+10$. In the former case, the output is at modulation $\sim 2^l$. Now either at least one of the inputs of the inner square bracket $[,]$ is at modulation $\geq 2^{l-10}$, and if one has modulation $\geq 2^{l+10}$, the other is at modulation $\geq 2^l$, or else the frequencies of the inputs of this square bracket are $\sim 2^l$. The former case is easy to treat by means of Bernstein's inequality and the

$X^{s,p,q}$ components of $S[k]$ (one gains exponentially in l). Assume w. l. o. g. that $k_2 \geq k_3$: if $l \leq k_2$, we compute, pulling out a $\nabla_{x,t}$ from the inner square bracket using its null-structure

$$\begin{aligned}
&||\nabla_{x,t}P_0Q_{l+O(1)}[P_{k_1}\psi_1\nabla^{-1}Q_l(1-I)[P_{k_2}Q_{l+O(1)}\psi_2, P_{k_3}\psi_3]]||_{\dot{X}_0^{-\frac{1}{2},-1,2}}\\
&\lesssim ||P_{k_1}\psi_1||_{L_t^\infty L_x^2}||P_{k_1}Q_l\nabla_{x,t}\nabla^{-1}[P_{k_2}Q_{l+O(1)}\nabla^{-1}\psi_2, P_{k_3}Q_{<l+O(1)}\psi_3]||_{L_t^2L_x^2}\\
&\lesssim 2^{-\frac{l}{2}}2^{l-k_1}2^{k_1-k_2}\prod_{i=1,3}||P_{k_i}\psi_i||_{L_t^\infty L_x^2}||P_{k_2}Q_{l+O(1)}\psi_2||_{\dot{X}_{k_2}^{0,\frac{1}{2},\infty}}.
\end{aligned}$$

Summing over $l > k_1+10$ yields the desired estimate. Further, if $l > k_2$, we estimate

$$\begin{aligned}
&||\nabla_{x,t}P_0Q_{l+O(1)}[P_{k_1}\psi_1\nabla^{-1}Q_l(1-I)[P_{k_2}Q_{l+O(1)}\psi_2, P_{k_3}\psi_3]]||_{\dot{X}_0^{-\frac{1}{2},-1,2}}\\
&\lesssim ||P_{k_1}\psi_1||_{L_t^\infty L_x^2}||P_{k_2}Q_{l+O(1)}R_0\psi_2||_{L_t^2L_x^2}||P_{k_3}\psi_3||_{L_t^\infty L_x^2}\\
&+||P_{k_1}\psi_1||_{L_t^\infty L_x^2}||P_{k_2}Q_{l+O(1)}\psi_2||_{L_t^2L_x^2}||P_{k_3}Q_{<l+O(1)}R_0\psi_3||_{L_t^\infty L_x^2},
\end{aligned}$$

whence

$$\begin{aligned}
&||\nabla_{x,t}P_0Q_{l+O(1)}[P_{k_1}\psi_1\nabla^{-1}Q_l(1-I)[P_{k_2}Q_{l+O(1)}\psi_2, P_{k_3}\psi_3]]||_{\dot{X}_0^{-\frac{1}{2},-1,2}}\\
&\lesssim 2^{-\frac{k_2}{2}}\prod_{i=1,3}||P_{k_i}\psi_i||_{L_t^\infty L_x^2}||P_{k_2}Q_l||_{\dot{X}_0^{-\frac{1}{2},1,2}}
\end{aligned}$$

Now one square-sums over l, obtaining an exponential gain in $-k_2$. The remaining cases when the modulations of ψ_2, ψ_3 are even bigger are more of the same.
The latter case is a bit more tricky: using schematic representation, we then have upon discarding the now unnecessary $(1-I)$ ($k_2 = k_3+O(1) = l+O(1)$, $l \geq k_1+10$)

$$\begin{aligned}
&\nabla_{x,t}P_0Q_{l+O(1)}[P_{k_1}\psi_1\nabla^{-1}Q_l[P_{k_2}Q_{<l-10}\psi_2,\ P_{k_3}Q_{<l-10}\psi_3]]\\
&=\sum_{\pm}\sum_{\kappa_{2,3}\in K_{-100},\,\mathrm{dist}(\kappa_2,\kappa_3)\sim 1}\nabla_{x,t}P_0Q_{l+O(1)}[P_{k_1}\psi_1\\
&\qquad\qquad\nabla^{-1}Q_l[P_{k_2,\kappa_2}Q^{\pm}_{<l-10}\psi_2,\ P_{k_3,\kappa_3}Q^{\pm}_{<l-10}\psi_3]],
\end{aligned}$$

where the $\pm$-signs need to match. Now decompose

$$P_{k_1}\psi_1 = P_{k_1}Q_{\geq k_1-200}\psi_1 + \sum_{\pm}\sum_{\kappa_1\in K_{-100}}P_{k_1,\kappa_1}Q^{\pm}_{<k_1-200}\psi_1$$

Substituting the first term on the right hand side, we have

$$\begin{aligned}
&||\nabla_{x,t}P_0Q_{l+O(1)}[P_{k_1}Q_{\geq k_1-200}\psi_1\nabla^{-1}Q_l[P_{k_2}Q_{<l-10}\psi_2,\ P_{k_3}Q_{<l-10}\psi_3]]||_{\dot{X}_0^{-\frac{1}{2},-1,2}}\\
&\lesssim 2^{-\frac{k_1}{2}}||P_{k_1}\psi_1||_{\dot{X}_{k_1}^{-\frac{1}{2},1,2}}||P_{k_i}\psi_i||_{L_t^\infty L_x^2}\lesssim 2^{-\frac{k_1}{2}}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]}
\end{aligned}$$

Next, substituting the sum, we note that every cap $\pm\kappa_1$ is at distance ~ 1 from either $\pm\kappa_2$ or $\pm\kappa_3$, where the $\pm$-signs are assigned according to whether the corresponding input is microsupported on the upper or lower half space. Abusing notation, we can replace the operator ∇^{-1} by 2^{-k_1}, given that it is a smooth convolution operator whose kernel has this L^1-mass, as well as the translation invariance of all Banach spaces used and the invariance of the Fourier support under translations. For example, we can estimate

$$\sum_{\pm}\sum_{\kappa_1\in K_{-100}}\sum_{\kappa_{2,3}\in K_{-100},\,\mathrm{dist}(\kappa_2,\kappa_3)\sim 1,\,\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 1}$$
$$\nabla_{x,t}P_0Q_{l+O(1)}[P_{k_1,\kappa_1}Q^{\pm}_{<k_1-200}\psi_1\nabla^{-1}Q_l[P_{k_2,\kappa_2}Q^{\pm}_{<l-10}\psi_2,\, P_{k_3,\kappa_3}Q^{\pm}_{<l-10}\psi_3]]||_{\dot{X}_0^{-\frac{1}{2},-1,2}}$$
$$\lesssim\sum_{\pm}\sum_{\kappa_1\in K_{-100}}\sum_{\kappa_{2,3}\in K_{-100},\,\mathrm{dist}(\kappa_2,\kappa_3)\sim 1,\,\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 1}$$
$$2^{-\frac{k_1}{2}}||P_{k_1,\kappa_1}Q^{\pm}_{<k_1-200}\psi_1||_{S[k_1,\pm\kappa_1]}||P_{k_2,\kappa_2}Q^{\pm}_{<l-10}\psi_2||_{S[k_2,\pm\kappa_2]}||P_{k_3}\psi_3||_{L_t^\infty L_x^2}$$

Using the definition of $S[k]$, this implies the desired estimate.

(1.2.b): *Output at modulation* $> 2^{l+10}$*.* In this case, the first input has to be at modulation $\geq 2^l$. This case is treated as at the beginning of case **(1.2.a)**.

(2): Now we assume $k_1\in[-10,10]$. We may as well assume that the inner square bracket $[,]$ is at frequency $< 2^{-10}$, since the opposite case doesn't offer anything new. Moreover, we also assume first that the output has modulation > 1, and the first input $P_{k_1}\psi_1$ has modulation < 1. Freezing the modulation of the output to size $\sim 2^l$, if at least one of the inputs of the inner $[,]$ has modulation $\geq 2^{l-10}$, we can again proceed as earlier. The only additional difficulty occurs when these inputs are both at modulation $< 2^{l-10}$, and hence at frequency $\sim 2^l$. In this case, utilizing schematic notation, observe that the following identity holds:

$$\nabla_{x,t}P_0Q_l[P_{k_1}Q_{<0}\psi_1\nabla^{-1}P_k[,]]=\sum_{\kappa\in K_k}\sum_{R\in C_{0,\kappa,k}}\nabla_{x,t}P_0Q_l\tilde{P}_R[P_{k_1}Q_{<0}\psi_1\nabla^{-1}P_k[,]]$$
$$=\sum_{\kappa\in K_k}\sum_{R\in C_{0,\kappa,k}}\nabla_{x,t}P_0Q_l\tilde{P}_R[\tilde{P}_{10R}Q_{<0}\psi_1\nabla^{-1}P_k[,]]$$

We are employing the notation used to define $S[k]$. In particular, $\tilde{P}_{10R}$ microlocalizes to a 'disc' which is the tenfold dilate of the 'disc' $R\in C_{0,\kappa,k}$. Next, we use Plancherel's theorem, and Bernstein's inequality to conclude that

$$||P_0Q_l[P_{k_1}\psi_1\nabla^{-1}P_k[,]]||_{L_t^2L_x^2}$$
$$\lesssim 2^k(\sum_{\kappa\in K_k}\sum_{R\in C_{0,\kappa,k}}||P_0Q_l\tilde{P}_R[\tilde{P}_{10R}Q_{<0}\psi_1\nabla^{-1}P_k[,]]||^2_{L_t^2L_x^1})^{\frac{1}{2}}$$
$$\lesssim 2^k(\sum_{\kappa\in K_k}\sum_{R\in C_{0,\kappa,k}}||[\tilde{P}_{10R}Q_{<0}\psi_1\nabla^{-1}P_k[,]]||^2_{L_t^2L_x^1})^{\frac{1}{2}}$$

Proceeding as before, we break up the inner square bracket into finitely many pieces, corresponding to further restricting the Fourier supports of the inputs to small angular sectors. As usual employing schematic notation, we have

$$\begin{aligned}&P_kQ_l[P_{k_2}Q_{<l-10}\psi_2,\, P_{k_3}Q_{<l-10}\psi_3]\\&=\sum_{\pm}\sum_{\kappa_{2,3}\in K_{-100},\mathrm{dist}(\kappa_2,\kappa_3)\sim 1}P_kQ_l[P_{k_2,\kappa_2}Q^{\pm}_{<l-10}\psi_2,\, P_{k_3,\kappa_3}Q^{\pm}_{<l-10}\psi_3]\end{aligned}$$

The summation being finite, we can restrict to a single pair $\kappa_{2,3}$, and we may also assume that $\mathrm{dist}(\pm 10\kappa_1, \pm\kappa_2)\sim 1$, the signs being determined according to whether the corresponding input is microlocalized to the upper or lower half-space $\tau >< 0$[27]. Invoking (14), we compute[28]

$$\begin{aligned}&||P_0Q_l[P_{k_1}\psi_1\nabla^{-1}P_k[P_{k_2,\kappa_2}Q^{\pm}_{<l-10},P_{k_3,\kappa_3}Q^{\pm}_{<l-10}\psi_3]]||_{L^2_tL^2_x}\\&\lesssim 2^k\sum_{\pm,\pm}(\sum_{\kappa\in K_k}\sum_{R\in C_{0,\kappa,k}}||[\tilde{P}_{10R}Q^{\pm}_{<0}\psi_1\nabla^{-1}P_k[P_{k_2,\kappa_2}Q^{\pm}_{<l-10},P_{k_3,\kappa_3}Q^{\pm}_{<l-10}\psi_3]]||^2_{L^2_tL^1_x})^{\frac{1}{2}}\\&\lesssim 2^{\frac{k}{2+}}(\sum_{\kappa\in K_k}\sum_{R\in C_{0,\kappa,k}}||\tilde{P}_{10R}Q^{\pm}_{<0}\psi_1||^2_{S[k_1,\pm 10\kappa_1]})^{\frac{1}{2}}||P_{k_2,\pm\kappa_2}\psi_2||_{S[k_2,\pm\kappa_2]}||P_{k_3}\psi_3||_{L^\infty_tL^2_x}\end{aligned}$$

One can sum this over $k<-10$, obtaining the desired inequality (from the definition of $S[k]$). Now assume that the output is at modulation ≤ 1. Freezing the frequency of the inner square bracket to size $\sim 2^k$, $k<-10$ and modulation $\sim 2^l$, $l>k+10$. Either the first input $P_{k_1}\psi_1$ is at modulation $<2^{l-10}$ and the output at modulation $\sim 2^l$, or the first input is at modulation $\geq 2^{l-10}$. In the former case, we have $l<O(1)$. Then either at least one of the inputs of the inner square bracket is at modulation $>2^{l-10}$, in which case one gains exponentially in $k-\max\{k_1,k_2,l\}$: assume w. l. o. g. that $k_2\geq k_3$. If $l\geq k_2$, we have

$$\begin{aligned}&||\nabla_{x,t}P_0Q_{<0}[P_{k_1}Q_{<l-10}\psi_1\nabla^{-1}P_kQ_l[P_{k_2}Q_{\geq l-10}\psi_2,P_{k_3}\psi_3]]||_{\dot{X}_0^{-1,-\frac{1}{2},2}}\\&\lesssim 2^{-l}2^{\min\{k,k_2,k_3\}}\prod_{i=1,3}||P_{k_i}\psi_i||_{L^\infty_tL^2_x}[||P_{k_2}\psi_2||_{\dot{X}_{k_2}^{0,\frac{1}{2},\infty}}+||P_{k_2}Q_{\geq k_2}\psi_2||_{\dot{X}_{k_2}^{-\frac{1}{2},1,2}}]\\&\lesssim 2^{\min\{k,k_2,k_3\}-l}\prod_{i=1,2,3}||P_{k_i}\psi_i||_{S[k_i]}\end{aligned}$$

One can sum here over k,l, obtaining an exponential gain in $-|k_2-k_3|$. On the other hand, when $l<k_2$, we pull out a $\nabla_{x,t}$ from the inner bracket, arriving at

$$\begin{aligned}&||\nabla_{x,t}P_0Q_{<0}[P_{k_1}Q_{<l-10}\psi_1\nabla_{x,t}\nabla^{-1}P_kQ_l[P_{k_2}Q_{\geq l-10}\nabla^{-1}\psi_2,P_{k_3}\psi_3]]||_{\dot{X}_0^{-1,-\frac{1}{2},2}}\\&\lesssim 2^{l-k}2^{-l}2^{2k-k_2}\prod_{i=1,2,3}||P_{k_i}\psi_i||_{S[k_i]}.\end{aligned}$$

[27] We also freeze the (space-time) Fourier support of $P_{k_1}Q_{<0}\psi_1$ to the upper or lower half space $\tau >< 0$.

[28] Again, we gloss over the tedious details of replacing $P_kQ_l\nabla^{-1}$ by a convolution operator and the inputs of $[,]$ by translates, to which the same microlocalizations and estimates apply.

Summing over $k_2 > l > k$ yields an exponential gain in $k - k_2$, whence we can sum over k, and $k_2 = k_3 + O(1)$. The case when $P_{k_3}\psi_3$ is at large modulation is handled analogously. Now assume both $P_{k_2}\psi_2$, $P_{k_3}\psi_3$ to be at modulation $< 2^{l-10}$. Then

$$\begin{aligned}
&||\nabla_{x,t} P_0 Q_{<0}[P_{k_1} Q_{<l-10}\psi_1 \nabla^{-1} P_k Q_l[,]]||_{N[0]} \\
&\lesssim 2^{-\frac{l}{2}} ||P_{k_1} Q_{<l-10}\psi_1||_{L_t^\infty L_x^2} ||P_k Q_l[,]||_{L_t^2 L_x^2} \\
&\lesssim 2^{\frac{k-l}{2+}} 2^{-\frac{|k_2-k_3|}{2}} \prod_{i=1}^{3} ||P_{k_i}\psi_i||_{S[k_i]},
\end{aligned}$$

as follows from lemma 6.2. Note that $l = k_2 + O(1)$. One can then sum over l, k. The case when the first input is at large modulation is handled analogously, placing the output into $L_t^1 L_x^2$.

(3): The case $k_1 < -10$ is easier and can be handled by the same methods. It is therefore left out. ■

Having disposed of this easy case, then, we proceed to the hard case when the inner square bracket $[,]$ is reduced to low modulation (i. e. the operator I applied in front). We state here the **main trilinear estimate**:

Proposition 6.11. *Let $\psi_1, \psi_2, \psi_3 \in \mathcal{S}(\mathbf{R^{2+1}})$. Also denote $I = \sum_{k\in\mathbf{Z}} P_k Q_{<k+10}$. Then we have the following inequalities for appropriate $\delta_i > 0$, $i = 1, 2$:*

$$\begin{aligned}
&||\partial^\beta P_0[R_\alpha P_{k_1}\psi_1 \triangle^{-1} \sum_{j=1}^{2} \partial_j I[R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_\beta P_{k_3}\psi_3 R_j P_{k_2}\psi_2]] \\
&+ \partial_\alpha P_0[R_\beta P_{k_1}\psi_1 \triangle^{-1} \sum_{j=1}^{2} \partial_j I[R^\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R^\beta P_{k_3}\psi_3]]||_{N[0]} \\
&\leq C 2^{\delta_1 \min\{-\min\{k_1,k_2,k_3\},0\}} \prod_i 2^{\delta_2 \min\{\max_{j\neq i}\{k_i, k_i-k_j\},0\}} \prod_l ||P_{k_l}\psi_l||_{S[k_l]},
\end{aligned} \tag{25}$$

$$\begin{aligned}
&||P_0 \partial^\beta [R_\beta P_{k_1}\psi_1 \triangle^{-1} \sum_j \partial_j I[R_\alpha P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R_\alpha P_{k_3}\psi_3]]||_{N[0]} \\
&\leq C 2^{\delta_1 \min\{-\min\{k_1,k_2,k_3\},0\}} \prod_i 2^{\delta_2 \min\{\max_{j\neq i}\{k_i, k_i-k_j\},0\}} \prod_l ||P_{k_l}\psi_l||_{S[k_l]}.
\end{aligned} \tag{26}$$

Remark: It appears that neither of the summands in the first inequality would satisfy a similar inequality on its own. We need to take advantage of a cancellation occuring between the two. This is in contrast with the 3-dimensional case, [19].

Proof : We treat the first inequality of the theorem in detail. The other one is easier, amenable to the same techniques and left for the interested reader. We shall mainly be concerned with analyzing destructive resonance cases as the other cases will follow more or less directly from Theorem 6.9. We split into three main cases,

corresponding to $k_1 > 10$, $k_1 \in [-10, 10]$, $k_1 < -10$.
(1): $k_1 > 10$. We shall treat the first summand of the first expression in the theorem. The other can be estimated identically (no cancellation needed yet). We commence by treating the case when the whole expression is microlocalized far away from the light cone: Freeze the modulation of the output to size 2^l, $l > 10$. Either $k_1 > l - 10$. Then we employ lemma 6.2 to place the output into $\dot{X}_0^{-\frac{1}{2},-1,1}$, achieving an exponential gain in $-|k_2 - k_3|$ as well as $-k_1$, which suffices to sum over l. Otherwise, the first input $P_{k_1} R_\beta \psi_1$ is at modulation $\sim 2^l$. In that case, we estimate

$$
\begin{aligned}
&||P_0 Q_l \partial^\beta [R_\beta P_{k_1} Q_{l+O(1)} \psi_1 \\
&\qquad \triangle^{-1} \sum_j \partial_j I[R_\alpha P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\alpha P_{k_3} \psi_3]]||_{\dot{X}_0^{-\frac{1}{2},-1,2}} \\
&\lesssim ||R_\beta P_{k_1} Q_{l+O(1)} \psi_1||_{L_t^2 L_x^2} \\
&\qquad ||\triangle^{-1} \sum_j \partial_j I[R_\alpha P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\alpha P_{k_3} \psi_3]||_{L_t^\infty L_x^2} \\
&\lesssim 2^{-\frac{k_1}{2}} 2^{-\frac{|k_2 - k_3|}{2}} ||P_{k_1} Q_l \psi_1||_{\dot{X}_{k_1}^{-\frac{1}{2},1,2}} \prod_{i=1}^2 ||P_{k_i} \psi_i||_{S[k_i]},
\end{aligned}
$$

where in the last line we used lemma 6.3. One can now square sum over l, obtaining the desired estimate. We can also easily place the output without the ∂^β outside into $L_t^M L_x^2$: provided the first input is microlocalized far away[29] from the light cone, we place it into $L_t^M L_x^2$ and the remainder of the expression into $L_t^\infty L_x^2$, using lemma 6.3. If the first input is microlocalized closely to the light cone, and in the remaining inputs R_0 doesn't fall on an input which is microlocalized far away from the light cone (i. e. 'elliptic'), we place the first input into $L_t^M L_x^N$ and the inner square bracket into $L_t^\infty L_x^{1+}$. Otherwise, one of the inputs of the inner square bracket is 'elliptic' and hit by R_0; we place it into $L_t^M L_x^2$ and the remaining inputs into $L_t^\infty L_x^2$. This settles the 'elliptic case', i. e. output at modulation $> 2^{10}$.
We next need to further distinguish between the possible frequency interactions inside the inner square bracket:
(1.1): *Assume in addition that $k_3 < 5$. In particular, $k_2 = k_1 + O(1)$.* Our strategy will be (cf. [18]) to reduce the modulations of the inputs and the expression in such fashion as to be able to take advantage of the inherent null-structure:
(1.1.a): *Reduce the modulation of the output to size $< 2^{(1-\epsilon)k_3}$, $\epsilon > 0$.* Reasoning as in the proof of the preceding theorem, we can easily reduce the modulation of the output to size < 1. Moreover, invoking lemma 6.2, we estimate

$$
\begin{aligned}
&||\partial^\beta P_0 Q_{[(1-\epsilon)k_3, 10]} [R_\alpha P_{k_1} \psi_1 \\
&\qquad \triangle^{-1} \sum_{j=1}^2 \partial_j I[R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_\beta P_{k_3} \psi_3 R_j P_{k_2} \psi_2]]||_{\dot{X}_0^{-1,-\frac{1}{2},1}}
\end{aligned}
$$

[29] I. e. its modulation is larger than its frequency.

$$\lesssim 2^{-\frac{(1-\epsilon)k_3}{2}}||R_\alpha P_{k_1}Q_{<O(1)}\psi_1||_{L_t^\infty L_x^2}$$
$$||\triangle^{-1}\sum_{j=1}^{2}\partial_j IP_{k_1+O(1)}[R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_\beta P_{k_3}\psi_3 R_j P_{k_2}\psi_2]||_{L_t^2 L_x^2}$$
$$\lesssim 2^{\frac{\epsilon}{2}k_3}2^{-k_1}\prod_{i=1}^{3}||P_{k_i}\psi_1||_{S[k_i]}.$$

This verifies the claims of the theorem for this case.
(1.1.b): *Reduce the modulation of the first input $P_{k_1}\psi_1$ to modulation $< 2^{(1-\epsilon)k_3-k_1}$.* This is achieved precisely as in the preceding case, placing the expression into $L_t^1 L_x^2$.
(1.1.c): *Under the above reductions, reduce the 2nd large frequency input $P_{k_2}\psi_2$ to modulation $< 2^{(1-\epsilon)k_3-k_2}$.* We replace the expression by the schematically written expression

$$"2^{-k_1}" P_0 Q_{<(1-\epsilon)k_3}[P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\psi_1\nabla_{x,t}\nabla^{-1}P_{k_3}\psi_3\nabla_{x,t}\nabla^{-1}P_{k_2}Q_{\geq(1-\epsilon)k_3-k_2}\psi_2].$$

We have replaced the operator $\nabla^{-1}P_{k_1+O(1)}I$ by "2^{-k_1}", which is made rigorous as usual by writing this operator in convolution form and substituting translates for certain inputs in the estimates below. The operators $\nabla_{x,t}\nabla^{-1}$ account for the Riesz type operators R_ν. In order to proceed, first throw an operator $Q_{\geq k_3}$ in front of $P_{k_3}\psi_3$. In that case, we can estimate the $L_t^1 L_x^2$-norm of the above by

$$\lesssim 2^{-k_1}||P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\psi_1||_{L_t^\infty L_x^2}||\nabla_{x,t}\nabla^{-1}P_{k_3}Q_{\geq k_3}\psi_3||_{L_t^2 L_x^\infty}$$
$$||\nabla_{x,t}\nabla^{-1}P_{k_2}Q_{\geq(1-\epsilon)k_3-k_2}\psi_2||_{L_t^2 L_x^2}$$
$$\lesssim 2^{\frac{\epsilon}{2}k_3}2^{-\frac{k_1}{2}}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]},$$

which is more than enough. Now apply an operator $Q_{<k_3}$ to the small-frequency input. Using the crude version of lemma 6.5 as well as lemma 6.4, we estimate the $N[0]$ norm of the above expression by

$$\lesssim 2^{-k_1}||P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\psi_1\nabla_{x,t}\nabla^{-1}P_{k_3}Q_{<k_3}\psi_3||_{\dot{X}_{k_1}^{0,\frac{1}{2},1}}$$
$$||\nabla_{x,t}\nabla^{-1}P_{k_2}Q_{\geq(1-\epsilon)k_3-k_2}\psi_2||_{\dot{X}_{k_2}^{0,-\frac{1}{2},1}}$$
$$\lesssim |k_3|2^{\epsilon k_3}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]},$$

which is again enough.
(1.1.d): *Under the above reductions, reduce the small frequency input to modulation $< 2^{(1-\epsilon)k_3+C}$*:[30] note the identity

[30]We denote generic large positive constants by C; they may change from line to line.

$$P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\psi_2 P_{k_3}Q_{\geq(1-\epsilon)k_3+C}\psi_3 = Q_{>(1-\epsilon)k_3}[P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\psi_2 P_{k_3}Q_{\geq(1-\epsilon)k_3+C}\psi_3]$$

Therefore, lemma 6.5 allows us to estimate this contribution by

$$\begin{aligned}
&\lesssim 2^{-\delta k_1}||P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\psi_1||_{S[k_1]} \\
&\qquad ||Q_{>(1-\epsilon)k_3}[P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\psi_2 P_{k_3}Q_{\geq(1-\epsilon)k_3+C}\psi_3]||_{\dot{X}_{k_2}^{0,-\frac{1}{2},1}} \\
&\lesssim 2^{\epsilon k_3}2^{-\delta k_1}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]}
\end{aligned}$$

Finally, we are in a position to **expand the null-structure: first letting the outer derivative fall on the first input** $P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\psi_1$, we use the identity

$$\begin{aligned}
&2\sum_{j=1}^{2}\triangle^{-1}\partial_j[R_\nu fR_jg - R_jfR_\nu g]\partial^\nu h \\
&= \sum_{j=1}^{2}\Box[\triangle^{-1}\partial_j[\nabla^{-1}fR_jg]h] - \sum_{j=1}^{2}\Box\triangle^{-1}\partial_j[\nabla^{-1}fR_jg]h \\
&- \sum_{j=1}^{2}\triangle^{-1}\partial_j[\nabla^{-1}fR_jg]\Box h - \nabla^{-1}f\Box((\nabla^{-1}g)h) \\
&+ \nabla^{-1}f\Box(\nabla^{-1}g)h + \nabla^{-1}f(\nabla^{-1}g)\Box h,
\end{aligned} \tag{27}$$

which was already used in [18]. We substitute the suitably microlocalized inputs and also microlocalize the whole expression as in the preceding. Then we need to estimate the following terms:

(1.1.e):

$$\begin{aligned}
&||\Box P_0Q_{<(1-\epsilon)k_3}(\nabla^{-1}[P_{k_3}Q_{<(1-\epsilon)k_3+C}\psi_3P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\nabla^{-1}\psi_2] \\
&\qquad\qquad P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\psi_1)||_{N[0]} \\
&\lesssim ||P_0Q_{<(1-\epsilon)k_3}(\nabla^{-1}[P_{k_3}Q_{<(1-\epsilon)k_3+C}\psi_3P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\nabla^{-1}\psi_2] \\
&\qquad\qquad P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\psi_1)||_{\dot{X}_0^{0,\frac{1}{2},1}}.
\end{aligned}$$

Now we use the following simple lemma:

Lemma 6.12. *Let $\psi_{1,2,3}\in\mathcal{S}(\mathbf{R}^{2+1})$. Then the following inequalities hold for $k_{1,2,3}$ as in the immediately preceding and $\epsilon>0$:*

$$||P_0[P_{k_1}\psi_1P_{k_2}\psi_2P_{k_3}\psi_3]||_{L_t^2L_x^{2+}} \lesssim 2^{(1+)k_1}2^{\frac{1+}{2}k_3}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]},$$

$$||P_0Q_{<O(1)}[P_{k_1}\psi_1P_{k_2}\psi_2P_{k_3}\psi_3]||_{\dot{X}_0^{0,\epsilon,1}} \lesssim 2^{\frac{3+}{4}k_3}2^{k_1}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]}.$$

Proof : The first inequality follows from lemma 3.1. The 2nd results from twofold application of lemma 6.4: Freeze the output to modulation $\sim 2^a$, $a < O(1)$. Similarly, freeze the inner expression $[P_{k_2}\psi_2 P_{k_3}\psi_3]$ to modulation $\sim 2^j$. We distinguish between the following two cases:

(1): $a > j + C$. We compute using lemma 6.4 twice:

$$||P_0 Q_a(P_{k_1}\psi_1 Q_j[P_{k_2}\psi_2 P_{k_3}\psi_3]||_{L_t^2 L_x^2}$$
$$\lesssim 2^{-\frac{a}{2}} 2^{\frac{a}{4+}} 2^{\min\{\frac{j-k_3}{4+},0\}} 2^{k_3} 2^{k_1} \prod_{i=1,2,3} ||P_{k_i}\psi_i||_{S[k_i]}.$$

The claim follows immediately upon summing over j, a (taking into account the factor $2^{\epsilon a}$).

(2): $a \leq j+C$. Again by lemma 6.4, as well as the improved Bernstein's inequality, we have

$$||P_0 Q_a(P_{k_1}\psi_1 Q_j[P_{k_2}\psi_2 P_{k_3}\psi_3]||_{L_t^2 L_x^2}$$
$$\lesssim 2^{\frac{a}{4}} ||P_{k_1}\psi_1||_{L_t^\infty L_x^2} ||Q_j[P_{k_2}\psi_2 P_{k_3}\psi_3]||_{L_t^2 L_x^2}$$
$$\lesssim 2^{\frac{a}{4}} 2^{-\frac{j}{2}} 2^{\min\{\frac{j-k_3}{4+},0\}} 2^{k_3} \prod_{i=1,2,3} ||P_{k_i}\psi_i||_{S[k_i]}.$$

This yields the inequality as before. ■

The desired estimate follows immediately. We next demonstrate how to deal with the fourth and sixth term of (27), the others being similar and simpler.
(1.1.f): For the fourth term, we use lemma 6.5 as well as lemma 6.4. We shall denote a small generic positive number by δ (which may change from line to line):

$$||P_0 Q_{<(1-\epsilon)k_3}(\nabla^{-1}\Box[P_{k_3}Q_{<(1-\epsilon)k_3+C}\nabla^{-1}\psi_3 P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\psi_1]$$
$$P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\psi_2)||_{N[0]}$$
$$\lesssim 2^{\delta k_3}||\nabla^{-1}\Box[P_{k_3}Q_{<(1-\epsilon)k_3+C}\nabla^{-1}\psi_3 P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\psi_1]||_{\dot{X}_{k_1}^{0,-\frac{1}{2},1}}$$
$$||P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\psi_2||_{S[k_2]}$$
$$\lesssim 2^{\delta k_3}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]}.$$

(1.1.g): For the sixth term, we freeze the modulation of $P_{k_1}Q_{<(1-\epsilon)k_3-k_1}\Box\nabla^{-1}\psi_1$ to modulation $\sim 2^j$. Then we estimate the following two contributions:

$$\begin{aligned}
&||P_0 Q_{<(1-\epsilon)k_3}(Q_{<\min\{j+C,(1-\epsilon)k_3+C\}}[\nabla^{-1}P_{k_3}Q_{<(1-\epsilon)k_3+C}\psi_3\\
&\qquad\qquad P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\psi_2]P_{k_1}Q_j\Box\nabla^{-1}\psi_1)||_{N[0]}\\
&\lesssim 2^{\delta k_3}||Q_{<\min\{j+C,(1-\epsilon)k_3+C\}}[\nabla^{-1}P_{k_3}Q_{<(1-\epsilon)k_3+C}\psi_3\\
&\qquad\qquad P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\psi_2]||_{\dot{X}_{k_1}^{0,\frac{1}{2},1}}||P_{k_1}Q_j\Box\nabla^{-1}\psi_1||_{\dot{X}_{k_1}^{0,-\frac{1}{2},\infty}}\\
&\lesssim 2^{\delta k_3}2^{\min\{\frac{j-k_3}{4+},0\}}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]}.
\end{aligned}$$

One can now sum over $j<(1-\epsilon)k_3-k_1$. The other contribution results from restricting the inner square bracket $[,]$ to modulation $\geq 2^{j+C}$: freeze this modulation to size $\sim 2^a$, $j+C\leq a<(1-\epsilon)k_3+C$:

$$\begin{aligned}
&||P_0 Q_{<(1-\epsilon)k_3}(Q_a[\nabla^{-1}P_{k_3}Q_{<(1-\epsilon)k_3+C}\psi_3\\
&\qquad\qquad P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\psi_2]P_{k_1}Q_j\Box\nabla^{-1}\psi_1)||_{N[0]}\\
&\lesssim 2^{\delta k_3}||Q_a[\nabla^{-1}P_{k_3}Q_{<(1-\epsilon)k_3+C}\psi_3\\
&\qquad\qquad P_{k_2}Q_{<(1-\epsilon)k_3-k_2}\psi_2]||_{\dot{X}_{k_2}^{0,-\frac{1}{2},1}}||P_{k_1}Q_j\Box\psi_1||_{\dot{X}_{k_1}^{0,\frac{1}{2},\infty}}\\
&\lesssim 2^{\delta k_3}2^{j-a}2^{\min\{\frac{a-k_3}{4+},0\}}\prod_{i=1}^{3}||P_{k_i}\psi_i||_{S[k_i]}.
\end{aligned}$$

Now sum over the suitable ranges of j, a in order to obtain the desired estimate.

Recalling the discussion in case **(1.1.d)**, we still have to consider the case when the outer derivative falls on the inner $Q_{\nu j}$ null-form. This case, however, can be expanded in a similar kind of null-form:

$$\begin{aligned}
&\sum_{j=1,2}\triangle^{-1}\partial^\nu\partial_j[R_\nu f R_j g-R_j f R_\nu g]h=h\sum_j\triangle^{-1}\partial_j\Box(\nabla^{-1}fR_jg)-\partial^\nu[\nabla^{-1}fR_\nu g]\\
&=h\sum_j\triangle^{-1}\partial_j\Box(\nabla^{-1}fR_jg)-\frac{1}{2}\Box(\nabla^{-1}f\nabla^{-1}g)h+\frac{1}{2}\Box\nabla^{-1}fg-\frac{1}{2}\nabla^{-1}f\Box\nabla^{-1}g
\end{aligned}$$

The terms on the right-hand side can be estimated in the exact same fashion. We are done with case **(1.1)**.

(1.2): Now we assume $k_1>10$, $k_1=k_2+O(1)$, $5\leq k_3\leq k_1$. The procedure is quite similar to case **(1.1)**. Using lemma 6.2, one reduces $P_{k_1}\psi_1$ to modulation $<2^{k_3-k_1}$. Similarly, one reduces the output to modulation <1. Next, we reduce the input $P_{k_2}\psi_2$ to modulation $<2^{k_3-k_2}$:

(1.2.a): *Reduce $P_{k_2}\psi_2$ to modulation $<C$.* Assume this input is at modulation 2^l, $l\geq C$. We may assume $l\leq k_2+O(1)$, the opposite case being simpler (in light of the operator I). Use schematic notation for the expression, as earlier, and shift the operator ∇^{-1} onto the first input, for convenience's sake:

$$
\begin{aligned}
&||P_0Q_{<0}(\nabla^{-1}P_{k_1}Q_{<k_3-k_1}\psi_1P_{k_3}\psi_3P_{k_2}Q_l\psi_2)||_{N[0]}\\
&\lesssim ||P_0Q_{<0}(Q_{l+O(1)}[\nabla^{-1}P_{k_1}Q_{<k_3-k_1}\psi_1P_{k_3}\psi_3]P_{k_2}Q_l\psi_2)||_{L_t^1L_x^2}\\
&\lesssim 2^{-\frac{l}{2}}2^{k_3-k_1}2^{\min\{\frac{l-k_3}{4+},0\}}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}.
\end{aligned}
$$

One can sum here over $l \geq C$.
(1.2.b): *Under previous reduction, reduce $P_{k_2}Q_{<C}\psi_2$ to modulation $< 2^{k_3-k_2}$.* We use the crude version of lemma 6.5:

$$
\begin{aligned}
&||P_0Q_{<0}(\nabla^{-1}P_{k_1}Q_{<k_3-k_1}\psi_1P_{k_3}\psi_3P_{k_2}Q_{[k_3-k_2,C]}\psi_2)||_{N[0]}\\
&= ||P_0Q_{<0}(Q_{<C}[\nabla^{-1}P_{k_1}Q_{<k_3-k_1}\psi_1P_{k_3}\psi_3]P_{k_2}Q_{[k_3-k_2,C]}\psi_2)||_{N[0]}\\
&\lesssim ||P_{k_1+O(1)}Q_{<C}[\nabla^{-1}P_{k_1}Q_{<k_3-k_1}\psi_1P_{k_3}\psi_3]||_{\dot{X}_{k_1}^{0,\frac{1}{2},1}}||P_{k_2}Q_{[k_3-k_2,C]}\psi_2||_{\dot{X}_{k_2}^{0,-\frac{1}{2},1}}\\
&\lesssim 2^{-\frac{k_3}{4+}}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}
\end{aligned}
$$

Finally, one reduces the small-frequency input $P_{k_3}\psi_3$ to modulation $< 2^C$, proceeding along the same lines. Next, one expands the null-structure, as in the preceding case. For example, one estimates:

$$
\begin{aligned}
&||P_0Q_{<0}(\Box\nabla^{-1}[P_{k_1}Q_{<k_3-k_1}\psi_1P_{k_3}Q_{<C}\nabla^{-1}\psi_3]P_{k_2}Q_{<k_3-k_2}\psi_2)||_{N[0]}\\
&\lesssim |k_3|||\Box\nabla^{-1}Q_{<C}[P_{k_1}Q_{<k_3-k_1}\psi_1P_{k_3}Q_{<C}\nabla^{-1}\psi_3]||_{\dot{X}_{k_1}^{0,-\frac{1}{2},1}}||P_{k_2}\psi_2||_{S[k_2]}\\
&\lesssim 2^{-\frac{k_3}{4+}}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}.
\end{aligned}
$$

We have used lemma 6.5. The remaining cases are monotonous repetitions of the same kinds of estimates, hence omitted.
(1.3): $k_1 > 10$, $k_3 = k_2 + O(1) \geq k_1$. This case doesn't offer anything new. One immediately reduces $P_{k_1}\psi_1$ to modulation $< O(1)$, using lemma 6.2. One pulls a derivative outside the inner square bracket, using the operator I applied in front of it, as well as the $Q_{\nu j}$-structure. Then one reduces both $P_{k_2}\psi_2$, $P_{k_3}\psi_3$ to modulation $< 2^{-k_2}$. Expanding the null-form, one proceeds as in the preceding case, and obtains an exponential gain in $-k_1$.
(2): $k_1 \in [-10, 10]$. The inner square bracket $[,]$ is then at frequency $< 2^{15}$, say. We freeze its frequency to dyadic value $\sim 2^k$, $k < 15$. We can easily reduce the output to modulation $< O(1)$, arguing as in the preceding case **(1)**. We are now interested in the following case:
(2.1): *High-high interactions within the inner square bracket: $k_2 = k_3+O(1) >> k$.* This case is harder than the previous ones on account of the fact that we need to gain exponentially in the difference $k - \min\{k_2, 0\}$ in order to be able to sum over $k < -10$. This cannot be achieved by means of lemma 6.2, for example. Indeed, we have to take advantage here of our modification of $S[k]$ with respect to earlier versions [18], [30], as well as a special cancellation between the summands in the

first expression (25) of the Proposition.
(2.1.a): *Output at modulation* $\geq 2^{k+\epsilon(k-\min\{k_2,0\})}$. *First input* $P_{k_1}\psi_1$ *at modulation* $\geq 2^{k+\epsilon(k-\min\{k_2,0\})}$. We may assume that the output is at modulation < 1, the opposite case being simple. Use the fact that the inner square bracket has very small L^∞-norm under these assumptions:

$$\begin{aligned}&||P_0Q_{[k+\epsilon(k-\min\{k_2,0\}),O(1)]}\partial^\nu[P_{k_1}Q_{\geq k+\epsilon(k-\min\{k_2,0\})}\psi_1\\ &\triangle^{-1}\sum_{j=1,2}\partial_jP_kQ_{<k+O(1)}[R_\nu P_{k_2}\psi_2R_jP_{k_3}\psi_3-R_jP_{k_2}\psi_2R_\nu P_{k_3}\psi_3]]||_{\dot{X}_0^{-1,-\frac{1}{2},1}}\\ &\lesssim 2^{-k-\epsilon(k-\min\{k_2,0\})}2^{2k-k_2}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}\end{aligned}$$

For $0<\epsilon<1$, one obtains the desired exponential gain in $k-\min\{k_2,0\}$. The 2nd summand in (25) is treated similarly.
(2.1.b): *Output at modulation* $\geq 2^{k+\epsilon(k-\min\{k_2,0\})}$. *First input* $P_{k_1}\psi_1$ *at modulation* $< 2^{k+\epsilon(k-\min\{k_2,0\})}$. We again treat the first summand in (25), the 2nd being handled similarly. **First, we let the outer derivative fall on the large frequency input** $P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_1$**.** Proceeding as in the previous number, we can also reduce $P_{k_i}\psi_i$, $i=2,3$, to modulation $<2^{2k-k_2}$. For simplicity's sake, denote $P_{k_1}Q^{\pm}_{<k+\epsilon(k-\min\{k_2,0\})}\psi_1=\phi_1^{\pm}$, $P_{k_i}Q^{\pm}_{<2k-k_2}\psi_i=\phi_i^{\pm}$, $i=2,3$ for the moment. Also, denote schematically

$$\triangle^{-1}\sum_{j=1,2}\partial_jP_kQ_{<k+O(1)}Q_{\nu j}[\phi_2^{\pm},\phi_3^{\pm}]=[\phi_2^{\pm},\phi_3^{\pm}]$$

Then, we use the following decomposition for our expression[31]:

$$\begin{aligned}&P_0Q_{[k+\epsilon(k-\min\{k_2,0\}),O(1)]}[\partial^\nu\phi_1^{\pm}[\phi_2^{\pm},\phi_3^{\pm}]]\\ &=\sum_{\pm,\pm,\pm}\sum_{\kappa_{1,2,3}\in K_{\frac{(1+\epsilon)(k-\min\{k_2,0\})}{2}},\mathrm{dist}(\pm\kappa_1,\pm\kappa_i)\leq 2^{(1+\epsilon)\frac{k-\min\{k_2,0\}}{2}+C},\,i=2,3}\\ &\qquad P_0Q_{[k+\epsilon(k-\min\{k_2,0\}),O(1)]}[\partial^\nu P_{k_1,\kappa_1}\phi_1^{\pm}[P_{k_2,\kappa_2}\phi_2^{\pm},P_{k_3,\kappa_3}\phi_3^{\pm}]]\\ &+\sum_{a>\frac{(1+\epsilon)(k-\min\{k_2,0\})}{2}+C}\sum_{\pm,\pm,\pm}\sum_{\kappa_{1,2,3}\in K_{a-10},\max\{\mathrm{dist}(\pm\kappa_1,\pm\kappa_i)\}\sim 2^a,\,i=2,3}\\ &\qquad P_0Q_{[k+\epsilon(k-\min\{k_2,0\}),O(1)]}[\partial^\nu P_{k_1,\kappa_1}\phi_1^{\pm}[P_{k_2,\kappa_2}\phi_2^{\pm},P_{k_3,\kappa_3}\phi_3^{\pm}]]\end{aligned}\tag{28}$$

We commence by estimating the last triple sum. Assume w. l. o. g. that $\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^a$. Note that provided we localize the Fourier support of $P_{k_2,\kappa_2}\phi_2^{\pm}$ further to a disc c_2 of radius 2^k, we can microlocalize $P_{k_3,\kappa_3}\phi_3^{\pm}$ to a disc c_3 of the same size such that $\mathrm{dist}(c_2,-c_3)\lesssim 2^k$. Moreover, freezing c_2, c_3 leaves only $O(1)$ many choices for $\kappa_{2,3}$. We now assume in addition to the preceding that $k_2<-10$. Unravelling the null-structure inside the inner square bracket, we move one operator ∇ outside. Replacing the operators $P_kQ_{<k+O(1)}\partial^j\partial_\nu\triangle^{-1}$ by convolution with

[31] We are fudging a bit below, since the operators P_{k_i,κ_i} appearing in the expression have to satisfy $P_{k_i,\kappa_i}P_{k_i}=P_{k_i}$. They have to be chosen differently than the operators used to define $\phi_i^{\pm}$.

a smooth kernel of L^1-mass $O(1)$, substituting translates for the inputs of $[,]$, and committing abuse of notation, we have to estimate schematically written expressions of the form

$$P_0 Q_{[k+\epsilon(k-k_2),O(1)]}[\partial^\nu P_{k_1,\kappa_1}\phi_1^\pm \nabla^{-1}P_{k_2,\kappa_2}P_{c_2}\phi_2^\pm P_{k_3,\kappa_3}P_{c_3}\phi_3^\pm]$$

If we now use the geometric observation in the proof of lemma 6.4, we have

$$P_{k_1,\kappa_1}\partial^\nu\phi_1^\pm\nabla^{-1}P_{k_2,\kappa_2}\phi_2^\pm = P_{k_1+O(1)}Q_{2a+k_2+O(1)}[P_{k_1,\kappa_1}\partial^\nu\phi_1^\pm\nabla^{-1}P_{k_2,\kappa_2}\phi_2^\pm]$$

We now use a simple variant of lemma 6.4, namely the following: let $k_{1,2}$, c_2 be as above. Let $C_{k_2,k-k_2}$ be a finitely overlapping covering of the frequency region $|\xi|\sim 2^{k_2}$ by means of discs c of size 2^k. Then

$$(\sum_{c_2\in C_{k_2,k-k_2}}||P_{k_1}\psi_1 P_{c_2}\psi_2||^2_{\dot{X}_0^{0,\frac{1}{2},\infty}})^{\frac{1}{2}} \lesssim 2^{k_2}2^{\frac{k-k_2}{2+}}\prod_{i=1,2}||P_{k_i}\psi_i||_{S[k_i]}.$$

Armed with this and using Cauchy-Schwarz as well as the definition of $S[k]$, we obtain

$$\begin{aligned}&\sum_{c_{2,3}\in C_{k_{2,3},k-k_{2,3}},\mathrm{dist}(c_2,-c_3)\lesssim 2^k}||P_0Q_{[k+\epsilon(k-k_2),O(1)]}[\partial^\nu P_{k_1,\kappa_1}\phi_1^\pm \\ &\qquad\qquad \nabla^{-1}P_{k_2,\kappa_2}P_{c_2}\phi_2^\pm P_{k_3,\kappa_3}P_{c_3}\phi_3^\pm]||_{N[0]} \\ &\lesssim 2^{-\frac{k+\epsilon(k-k_2)}{2}}2^k(\sum_{c_3\in C_{k_3,k-k_3}}||P_{k_3,\kappa_3}P_{c_3}\phi_3^\pm||^2_{L_t^\infty L_x^2})^{\frac{1}{2}} \\ &\qquad (\sum_{c_2\in C_{k_2,k-k_2}}||\partial^\nu P_{k_1,\kappa_1}\phi_1^\pm\nabla^{-1}P_{k_2,\kappa_2}P_{c_2}\phi_2^\pm||^2_{L_t^2L_x^2})^{\frac{1}{2}} \\ &\lesssim 2^{-\frac{k+\epsilon(k-k_2)}{2}-a-\frac{k_2}{2}}2^{k+\frac{k-k_2}{2+}}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}\end{aligned}$$

One can sum over $a>\frac{(1+\epsilon)(k-k_2)}{2}+C$, provided $\epsilon<<\frac{1}{2}$, getting the desired exponential gain in $k-k_2$. Next, assume that $k_2\in[-10,10]$. In this case there may be destructive resonance between the first two inputs $\partial^\nu P_{k_1,\kappa_1}\phi_1^\pm$, $\nabla^{-1}P_{k_2,\kappa_2}P_{c_2}\phi_2^\pm$. The angular separation condition ensures that the frequency of their product is at least $\sim 2^a$. Freezing this frequency to dyadic value $\sim 2^r$, and also freezing the modulation of this product to dyadic size $\sim 2^j$, the geometric observation in the proof of lemma 6.4 implies

$$2^{\frac{\max\{j,k+\epsilon(k-k_2)\}+r}{2}-k_2}\geq 2^{a+O(1)},$$

which yields $j\geq 2a-r+2k_2+O(1)$. Now we use another simple variant of lemma 6.4, namely that under the present assumptions on $k_{1,2}, k, r$, we have

$$(\sum_{c\in C_{k_2,k}} ||P_r[P_{k_1}\psi_1 P_c\psi_2]||^2_{\dot{X}_r^{0,\frac{1}{2},\infty}})^{\frac{1}{2}} \lesssim 2^{k_2} 2^{\frac{k-r}{2+}} \prod_i ||P_{k_i}\psi_i||_{S[k_i]}$$

Plugging this in, and using Cauchy-Schwarz and the definition of $S[k]$ as before, we estimate this case by

$$\lesssim 2^{-\frac{k+\epsilon(k-k_2)}{2}} 2^{-a+\frac{r}{2}} 2^k 2^{\frac{k-r}{2+}} \prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$$

This can be summed over the above indicated ranges of r, a to yield an exponential gain in $k-k_2$, provided $\epsilon << \frac{1}{2}$. When $k_2 > 10$, one simply uses

$$P_{k_1,\kappa_1}\partial^\nu \phi_1^{\pm} \nabla^{-1} P_{k_2,\kappa_2} P_{c_2}\phi_2^{\pm} = P_{k_2+O(1)} Q_{2a+O(1)}[P_{k_1,\kappa_1}\partial^\nu \phi_1^{\pm} \nabla^{-1} P_{k_2,\kappa_2} P_{c_2}\phi_2^{\pm}]$$

Then one proceeds exactly as above, obtaining an exponential gain in k. We now proceed to the first sum in (28), corresponding to the case when $\pm\kappa_1$ is closely aligned with both $\pm\kappa_{2,3}$. We suppress the operator $P_k Q_{<k}\partial_j \triangle^{-1}$ applied to the inner square bracket $[,]$, keeping in mind that it costs 2^{-k} at the end of the day, and recall what $[,]$ stands for. Freezing $c_{2,3}$ etc. for the moment, we obtain a sum of two schematic expressions of the form

$$P_0 Q_{[k+\epsilon(k-\min\{k_2,0\}),O(1)]}[P_{k_1,\kappa_1}\partial^\nu \phi_1^{\pm} R_\nu P_{k_2,\kappa_2} P_{c_2}\phi_2^{\pm} P_{k_3,\kappa_3} P_{c_3}\phi_3^{\pm}]$$

Assuming for the moment that $k_2 < -10$, we can rewrite the above as

$$P_0 Q_{[k+\epsilon(k-k_2),O(1)]}[P_{k_1+O(1)} Q_{<k+\epsilon(k-k_2)+O(1)}[P_{k_1,\kappa_1}\partial^\nu \phi_1^{\pm} \\ R_\nu P_{k_2,\kappa_2} P_{c_2}\phi_2^{\pm}] P_{k_3,\kappa_3} P_{c_3}\phi_3^{\pm}]$$

Our next step of course consists in unravelling the Q_0-null-structure inside the inner bracket, using $2\partial_\nu u \partial^\nu v = \Box(uv) - \Box u v - u\Box v$. For example, we can estimate

$$\begin{aligned} &\sum_{c_{2,3}\in C_{k_{2,3},k-k_{2,3}},\, \mathrm{dist}(c_2,-c_3)\lesssim 2^k} ||P_0 Q_{[k+\epsilon(k-k_2),O(1)]}[P_{k_1+O(1)} Q_{<k+\epsilon(k-k_2)+O(1)} \\ &\qquad \Box[P_{k_1,\kappa_1}\phi_1^{\pm} \nabla^{-1} P_{k_2,\kappa_2} P_{c_2}\phi_2^{\pm}] P_{k_3,\kappa_3} P_{c_3}\phi_3^{\pm}]||_{N[0]} \\ &\lesssim 2^k \sum_{c_{2,3}\in C_{k_{2,3},k-k_{2,3}},\, \mathrm{dist}(c_2,-c_3)\lesssim 2^k} ||[P_{k_1,\kappa_1}\phi_1^{\pm} \\ &\qquad \nabla^{-1} P_{k_2,\kappa_2} P_{c_2}\phi_2^{\pm}]||_{\dot{X}_0^{0,\frac{1}{2},1}} ||P_{k_3,\kappa_3} P_{c_3}\phi_3^{\pm}||_{L_t^\infty L_x^2} \\ &\lesssim 2^k 2^{\frac{k-k_2}{2+}} \prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}. \end{aligned}$$

This is enough to absorb the 2^{-k} which we still have to pay for discarding the operator $P_kQ_{<k}\nabla^{-1}$ before. The remaining terms of the null-form expansion, as well as the other cases $k_2 \in [-10, 10]$, $k_2 > 10$ are handled in exact analogy to the preceding and omitted. **Now we let the outer derivative fall on the low-frequency inner square bracket** $[,]$**.** Using the same notation as before, we need to estimate the following expression:

$$\begin{aligned}
&P_0Q_{[(1+\epsilon)k,O(1)]}[\phi_1\triangle^{-1}\sum_{j=1,2}\partial_j\partial^\nu P_kQ_{<k+O(1)}[R_\nu\phi_2R_jP_{k_3}\phi_3 - R_j\phi_2R_\nu\phi_3]\\
&= P_0Q_{[(1+\epsilon)k,O(1)]}[\phi_1\triangle^{-1}\sum_{j=1,2}\partial_j\Box P_kQ_{<k+O(1)}[\nabla^{-1}\phi_2R_j\phi_3]]\\
&\qquad\qquad - P_0Q_{\geq(1+\epsilon)k_3}[\phi_1P_kQ_{<k+O(1)}\partial^\nu[\nabla^{-1}\phi_2R_\nu\phi_3]]
\end{aligned}$$

Everything here is straightforward to estimate. For example, using lemma 6.4, we get for the first summand:

$$\begin{aligned}
&||P_0Q_{[(1+\epsilon)k,O(1)]}[\phi_1\triangle^{-1}\sum_{j=1,2}\partial_j\Box P_kQ_{<k+O(1)}[\nabla^{-1}\phi_2R_j\phi_3]]||_{\dot{X}_0^{-1,-\frac{1}{2},1}}\\
&\lesssim 2^{-\frac{(1+\epsilon)k}{2}}||\phi_1||_{L_t^\infty L_x^2}||\sum_{j=1,2}\triangle^{-1}\partial_j\Box P_kQ_{<k+O(1)}[\nabla^{-1}\phi_2R_j\phi_3]||_{L_t^2L_x^\infty}\\
&\lesssim 2^{-\frac{(1+\epsilon)k}{2}}2^{\frac{k}{2}}2^k\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}
\end{aligned}$$

The 2nd summand is estimated similarly be expanding the inner bracket, using a simple null-form identity. This concludes step **(2.1.b)**.

(2.1.c): *Output at modulation* $< 2^{k+\epsilon(k-\min\{k_2,0\})}$, *first input* $P_{k_1}\psi_1$ *at modulation* $\geq 2^{k+\epsilon(k-\min\{k_2,0\})+C}$. It is easy to see that we can immediately reduce the two inputs $P_{k_{2,3}}\psi_{2,3}$ to modulation $< 2^{k+\epsilon(k-k_2)}$. Pulling out one operator ∇ from the inner square bracket $[,]$ (using its null-structure) and suppressing the operator $\triangle^{-1}\partial_j\nabla P_kQ_{<k+O(1)}$, we represent this case schematically (and as usual abusing notation):

$$\begin{aligned}
&\sum_{c_{2,3}\in C_{k_2,k-k_2},\,\mathrm{dist}(c_2,-c_3)\lesssim 2^k}\\
&P_0Q_{<k+\epsilon(k-\min\{k_2,0\})}([\partial^\nu P_{k_1}Q_{\geq k+\epsilon(k-\min\{k_2,0\})+C}\psi_1\\
&\qquad\qquad\nabla^{-1}P_{c_2}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_2]P_{c_3}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_3)
\end{aligned}$$

We observe that we may always avoid destructive resonance in the inner square bracket here: simply break $P_{k_1}\partial^\nu\phi_1$ into finitely many pieces microlocalized along angular sectors, and switch the last two inputs $P_{c_2}\psi_2$, $P_{c_3}\psi_3$, if c_2 is opposite to the corresponding angular sector. We claim that we can throw an operator $Q_{\geq k+\epsilon(k-\min\{k_2,0\})}$ in front of $[,]$ without altering the output. Indeed, suppose we do the opposite and apply $Q_{<k+\epsilon(k-\min\{k_2,0\})}$. Using the geometric observation employed many times before, we have (omitting summations over $c_{2,3}$ for now)[32]

[32]We denote by $\kappa(c)$ the angular sector associated with a disc c.

$$P_0Q_{<k+\epsilon(k-\min\{k_2,0\})}(Q_{<k+\epsilon(k-\min\{k_2,0\})}[\partial^\nu P_{k_1}Q_{\geq k+\epsilon(k-\min\{k_2,0\})+C}\psi_1 \\ \nabla^{-1}P_{c_2}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_2]P_{c_3}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_3)$$
$$= \sum_{\pm,\pm,\pm}\sum'_{\kappa_1\in K_{\frac{(1+\epsilon)(k-\min\{k_2,0\})}{2}-\max\{k_2,0\}+C-10}}$$
$$P_0Q_{<k+\epsilon(k-\min\{k_2,0\})}(P_{\tilde{k},\kappa_1}Q^{\pm}_{<k+\epsilon(k-\min\{k_2,0\})}[\partial^\nu P_{k_1}Q^{\pm}_{\geq k+\epsilon(k-\min\{k_2,0\})+C}\psi_1 \\ \nabla^{-1}P_{c_2}Q^{\pm}_{<k+\epsilon(k-\min\{k_2,0\})}\psi_2]P_{c_3}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_3),$$

where $\sum'$ is only extended over those caps κ_1 satisfying

$$\mathrm{dist}(\pm\kappa_1,\pm\kappa(c_2))\geq 2^{\frac{(1+\epsilon)(k-\min\{k_2,0\})}{2}-\max\{k_2,0\}+C},$$

and $\tilde{k}=\max\{k_2,0\}+O(1)$. Similarly, we have

$$P_0Q_{<k+\epsilon(k-\min\{k_2,0\})}(Q_{<k+\epsilon(k-\min\{k_2,0\})}[\partial^\nu P_{k_1}Q_{\geq k+\epsilon(k-\min\{k_2,0\})+C}\psi_1 \\ \nabla^{-1}P_{c_2}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_2]P_{c_3}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_3)$$
$$= \sum_{\pm,\pm,\pm}\sum''_{\kappa_2\in K_{\frac{(1+\epsilon)(k-\min\{k_2,0\})}{2}-\max\{k_2,0\}-10}}$$
$$P_0Q_{<k+\epsilon(k-\min\{k_2,0\})}(P_{\tilde{k},\kappa_2}Q^{\pm}_{<k+\epsilon(k-\min\{k_2,0\})}[\partial^\nu P_{k_1}Q^{\pm}_{\geq k+\epsilon(k-\min\{k_2,0\})+C}\psi_1 \\ \nabla^{-1}P_{c_2}Q^{\pm}_{<k+\epsilon(k-\min\{k_2,0\})}\psi_2]P_{c_3}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_3),$$

where $\sum''$ is extended over those caps κ_2 satisfying

$$\mathrm{dist}(\pm\kappa_2,\pm\kappa(c_3))\leq 2^{\frac{(1+\epsilon)(k-\min\{k_2,0\})}{2}-\max\{k_2,0\}+O(1)}.$$

Since we already know that $\mathrm{dist}(\pm\kappa(c_2),\pm\kappa(c_3))\leq 2^{k-k_2+O(1)}$, where the $\pm$ signs are determined as usual according to the situation of the (space-time) Fourier support, we conclude that $P_{\tilde{k},\kappa_1}P_{\tilde{k},\kappa_2}=0$, hence the whole contribution vanishes. We now use Cauchy-Schwarz and the following simple variant of lemma 6.5: Let $F_c\in\mathcal{S}(\mathbf{R}^{2+1})$, indexed by a set of discs $c\in C_{k_3,k-k_3}$, and also $\psi\in\mathcal{S}(\mathbf{R}^{2+1})$; then, for $k_{2,3}$, k as before, we have the inequality

$$\sum_{c\in C_{k_3,k-k_3}}||P_0[P_{k_2+O(1)}F_cP_c\psi]||_{N[0]}\lesssim 2^{\min\{k_3,0\}}2^{\frac{k-\min\{k_3,0\}}{2+}}(\max\{k_3,0\}+1) \\ (\sum_{c\in C_{k_3,k-k_3}}||P_{k_2+O(1)}F_c||^2_{\dot{X}^{0,-\frac{1}{2},1}_{k_2}})^{\frac{1}{2}}||P_{k_3}\psi_3||_{S[k_3]}.$$

This follows by going through the proof of lemma 6.5 for fixed ψ_c, F_c, carefully keeping track of the modulations, and then summing over c, using Cauchy-Schwarz as well as the definition of $S[k]$ and Plancherel. We can now estimate

$$||\sum_{c_{2,3}\in C_{k_2,k-k_2},\mathrm{dist}(c_2,-c_3)\lesssim 2^k} P_0Q_{<k+\epsilon(k-\min\{k_2,0\})}(Q_{\geq k+\epsilon(k-\min\{k_2,0\})}[\partial^\nu P_{k_1}Q_{\geq k+\epsilon(k-\min\{k_2,0\})+C}\psi_1 \nabla^{-1}P_{c_2}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_2]P_{c_3}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_3)||_{N[0]}$$

$$\lesssim 2^{\frac{k-\min\{k_2,0\}}{2+}}2^{\min\{k_3,0\}}(\max\{k_3,0\}+1)||P_{k_3}\psi_3||_{S[k_3]} \sum_{a,b\geq k+\epsilon(k-\min\{k_2,0\})}(\sum_{c_2\in C_{k_2,k-k_2}}||Q_a[\partial^\nu P_{k_1}Q_b\psi_1 \nabla^{-1}P_{c_2}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_2]||^2_{\dot{X}^{0,-\frac{1}{2},\infty}_{\max\{k_2,0\}}})^{\frac{1}{2}}$$

But we can easily estimate

$$(\sum_{c_2\in C_{k_2,k-k_2}}||Q_a[\partial^\nu P_{k_1}Q_b\psi_1\nabla^{-1}P_{c_2}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_2]||^2_{\dot{X}^{0,-\frac{1}{2},\infty}_{\max\{k_2,0\}}})^{\frac{1}{2}}$$
$$\lesssim 2^{k-k_2}|k-k_2|2^{-\frac{a+b}{2}}||P_{k_1}\psi_1||_{\dot{X}^{0,\frac{1}{2},\infty}_{k_1}}||P_{k_2}\psi_2||_{S[k_2]},$$

and the desired estimate follows easily from this provided we choose $\epsilon << \frac{1}{2}$.
(2.1.d): *Both output and first input at modulation* $< 2^{k+\epsilon(k-\min\{k_2,0\})}$, $< 2^{k+\epsilon(k-\min\{k_2,0\})+C}$*, respectively.* In this case, we expand the null-structure. We shall use the following identity, which is subtly different than (27):

$$\begin{aligned}
&2(\partial^\nu f\sum_j \triangle^{-1}\partial_j[R_\nu gR_jh-R_jgR_\nu h])+f\sum_j\triangle^{-1}\partial_j\partial^\nu[R_\nu gR_jh-R_jgR_\nu h]\\
&=\Box[f\sum_j\triangle^{-1}\partial_j(\nabla^{-1}gR_jh)]-(\Box f)\sum_j\triangle^{-1}\partial_j(\nabla^{-1}gR_jh)\\
&\qquad-2\partial^\nu f\nabla^{-1}gR_\nu h-f\partial^\nu(\nabla^{-1}gR_\nu h)
\end{aligned}\tag{29}$$

Plugging in the suitably microlocalized inputs as before, we need to estimate the following terms:
(2.1.e):

$$\Box P_0Q_{<k+\epsilon(k-\min\{k_2,0\})}(P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_1 \nabla^{-1}P_kQ_{<k+O(1)}[\nabla^{-1}P_{k_2}\psi_2P_{k_3}\psi_3])$$

We estimate this by means of the following lemma, which is a refinement of lemma 6.12 and proved in exactly the same way:

Lemma 6.13. *Let* $k_1,k_2,k_3>>r$ *be integers. Then for every* $\epsilon>0$*, we have*

$$||P_{k_1}Q_{<r}\psi_1P_rQ_{<r+O(1)}[\nabla^{-1}P_{k_2}\psi_2P_{k_3}\psi_3]||_{\dot{X}^{0,,\epsilon,1}_{k_1}}\lesssim 2^{(\frac{1}{2}+\epsilon)r}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}$$

Armed with this, we can now estimate

$$||\Box P_0 Q_{<k+\epsilon(k-\min\{k_2,0\})}(P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})+C}\psi_1 \nabla^{-1}P_kQ_{<k+O(1)}[\nabla^{-1}P_{k_2}\psi_2 P_{k_3}\psi_3])||_{\dot{X}_0^{0,-\frac{1}{2},1}}$$

$$\lesssim 2^{\frac{\epsilon}{2}(k-\min\{k_2,0\})}\prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$$

(2.1.f):

$$P_0 Q_{<k+\epsilon(k-\min\{k_2,0\})}[P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})+C}\Box\psi_1 P_kQ_{<k+O(1)}\nabla^{-1}[\nabla^{-1}P_{k_2}\psi_2 P_{k_3}\psi_3]]$$

We can estimate this using lemma 6.5 as well as lemma 6.4:

$$||P_0 Q_{<k+\epsilon(k-\min\{k_2,0\})}[P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})+C}\Box\psi_1 P_kQ_{<k+O(1)}\nabla^{-1}[\nabla^{-1}P_{k_2}\psi_2 P_{k_3}\psi_3]]||_{N[0]}$$

$$\lesssim \sum_{j<k+\epsilon(k-\min\{k_2,0\})+C} 2^{\delta(j-k)}||\Box P_{k_1}Q_j\psi_1||_{\dot{X}_{k_1}^{0,-\frac{1}{2},\infty}} ||P_kQ_{<k+O(1)}\nabla^{-1}[\nabla^{-1}P_{k_2}\psi_2 P_{k_3}\psi_3]||_{\dot{X}_k^{1,\frac{1}{2},1}}$$

$$\lesssim 2^{\epsilon\delta(k-\min\{k_2,0\})}\prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$$

(2.1.g):

The remaining terms of the null-form expansion are much easier: indeed, the k here plays no role at all. For the third term, we can retrace the steps that lead to the introduction of the localizers $Q_{<k+\epsilon(k-\min\{k_2,0\})}$ etc. and then apply Tao's theorem 6.9. As for the last term of the null-form expansion, we rewrite it as

$$P_0 Q_{<k+\epsilon(k-\min\{k_2,0\})}[P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_1 P_kQ_{<k+O(1)}\partial^\nu[\nabla^{-1}P_{k_2}\psi_2 R_\nu P_{k_3}\psi_3]]$$

We first reduce $P_{k_{2,3}}\psi_{2,3}$ to modulation $< 2^{k+C}$, which is straightforward, and then expand the Q_0-type null-structure. For example, suppressing the operator $P_kQ_{<k+O(1)}$, we have

$$||P_0 Q_{<k+\epsilon(k-\min\{k_2,0\})}[P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_1 \nabla^{-1}P_{k_2}Q_{<k+C}\psi_2 P_{k_3}Q_{<k+C}\Box\nabla^{-1}\psi_3||_{N[0]}$$

$$\lesssim \sum_{j<k+C} ||P_0 Q_{<k+\epsilon(k-\min\{k_2,0\})}[Q_{<j}(P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_1 \nabla^{-1}P_{k_2}Q_{<k+C}\psi_2)P_{k_3}Q_j\Box\nabla^{-1}\psi_3||_{N[0]}$$

$$+\sum_{j<k+C}||P_0Q_{<k+\epsilon(k-\min\{k_2,0\})}[Q_{\geq j}(P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_1 \nabla^{-1}P_{k_2}Q_{<k+C}\psi_2)P_{k_3}Q_j\Box\nabla^{-1}\psi_3||_{N[0]}$$

The first of the last two summands is estimated by means of the crude version of lemma 6.5:

$$\sum_{j<k+C}||P_0Q_{<k+\epsilon(k-\min\{k_2,0\})}[Q_{<j}(P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_1 \nabla^{-1}P_{k_2}Q_{<k+C}\psi_2)P_{k_3}Q_j\Box\nabla^{-1}\psi_3||_{N[0]}$$
$$\lesssim\sum_{j<k+C}||Q_{<j}(P_{k_1}Q_{<k+\epsilon(k-\min\{k_2,0\})}\psi_1 \nabla^{-1}P_{k_2}Q_{<k+C}\psi_2)||_{\dot{X}^{0,\frac{1}{2},1}_{\max\{k_2,0\}}}||P_{k_3}Q_j\Box\nabla^{-1}\psi_3||_{\dot{X}^{0,-\frac{1}{2},1}_{k_3}}$$
$$\lesssim\sum_{j<k+C}2^{\frac{j-\min\{k_2,0\}}{4+}}2^{-\max\{k_2,0\}}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]},$$

which is as desired. The 2nd sum in the preceding is estimated similarly, as are the remaining terms of the null-form expansion. We are done with the case **(2.1)**.

(2.2): *Again* $k_1\in[-10,10]$*, but high-low or low-high interactions in the inner square bracket, i. e.* $k_2=k+O(1)$ *or* $k_3=k+O(1)$. This follows directly from Theorem 6.9, or also as in [18]. This concludes case **(2)**.

(3): $k_1<-10$. This case is much simpler than the preceding as far as possible destructive resonance phenomena are concerned. Indeed, the case when destructive resonance occurs in the inner square bracket ($k_2=k_3+O(1)>O(1)$) is treated by the exact same methods, as in case **(1)**. The difficulty occurs in treating the 2nd summand in (25) when k_2, k_3 are widely different, but this was achieved by Tao in Theorem 6.9. This concludes the proof of the Proposition 6.11. ■

7. Quintilinear terms.

We now attack the quintilinear error terms shortly discussed in section 5.2. Our procedure shall be roughly as follows: for a schematic expression

$$\nabla_{x,t}[\psi_1\nabla^{-1}[\psi_2\nabla^{-1}(\psi_3\nabla^{-1}(\psi_4\psi_5))]],$$

we (i) frequency-localize the inputs ψ_i to dyadic frequencies $\sim 2^{k_i}$, and the output to frequency ~ 1. Then we (ii) keep track of possible destructive resonance phenomena (many cases!). This may force us to frequency-localize larger constituents of the above expression, for example $\nabla^{-1}[\psi_2 \dots]$. Finally, having estimated the frequency-localized expressions, we need to be able to sum over all frequency-parameters. In the end, we typically pair off the indices k_i in some fashion, obtaining for example exponential gains in $-|k_2-k_3|$, $-|k_4-k_5|$. This allows us to reduce l^1 summation to l^2 summation via Cauchy-Schwarz. We then also need to obtain an exponential gain in some $-|k_i|$, i. e. the difference of an input frequency and the output frequency, in order to retrieve the original frequency envelope. On account of the multilinear nature of the expressions, one is tempted to invoke some sort of algebra estimates. Unfortunately, every extra factor ψ entails an extra operator ∇^{-1} falling on a combination of inputs. This renders the treatment extraordinarily cumbersome. If one had sharp improved Strichartz type norms available, the sequel could be massively simplified. This is somewhat analogous to the situation in $3+1$ dimensions, where the failure of the endpoint Strichartz estimate caused all the complications for the trilinear terms. In the present situation, though, the problem is not the lack of an $L_t^4 L_x^\infty$-Strichartz estimate, but rather the author's inability to build this norm into the spaces $S[k]$, due to a logarithmic divergence. The author apologizes for the piecemeal estimates to follow, which are certainly amenable to simplification, if only by redesigning the spaces employed. Returning to the quintilinear estimates in detail, the 2nd of these recorded in section 5 is significantly more difficult, and we address it first. Recall that it has the following schematic form:

$$\nabla_{x,t}[\psi \nabla^{-1}(R_\beta \psi \chi_\nu)],$$

where χ_ν is defined as in section 2, and not both $\beta = 0$, $\nu = 0$. Our first task consists in restricting $R_\beta \psi$ to 'hyperbolic microsupport'. We relegate this simple but tedious step to a technical appendix. Having accomplished this, we apply dynamic separation to χ_ν, decomposing it into a $Q_{\nu j}$-type null-form and elliptic error terms. Focusing on the quintilinear terms containing $Q_{\nu j}$, we need to reduce $Q_{\nu j}$ to hyperbolic microsupport. This is again a tedious technical step relegated to the appendix.

It turns out that the resulting expression is in many cases amenable to estimation by means of improved type Strichartz norms introduced in section 6. Indeed, using the operator $I = \sum_{k \in \mathbf{Z}} P_k Q_{<k+100}$, we have the following

Lemma 7.1. *Let $\psi_i \in \mathcal{S}(\mathbf{R}^{2+1})$, $i = 1, \dots 5$ satisfy $||P_k \psi_i||_{S[k_i]} \leq c_k$ for a 'sufficiently flat' frequency envelope. Then the following inequality holds:*

$$||\nabla_{x,t} P_0[\psi_1 \nabla^{-1} P_{>-10}(\psi_2 \nabla^{-1}(\psi_3 \nabla^{-1} Q_{\nu j} I(\psi_4, \psi_5))]||_{N[0]} \lesssim c_0$$

Proof : First consider the contribution when the output is reduced to modulation > 1. One then uses lemma 6.2 to place it into $\dot{X}_0^{-\frac{1}{2},-1,2}$. Similarly, it is straightforward to place the output without the operator $\nabla_{x,t}$ in front into $L_t^M L_x^2$. Now restrict the output to modulation < 1, whence we can discard the operator $\nabla_{x,t}$ in front. We frequency-localize the expression to

$$P_0[P_{k_1}\psi_1 \nabla^{-1} P_{a_1}(P_{k_2}\psi_2 \nabla^{-1} P_{a_2}(P_{k_3}\psi_3 \nabla^{-1} P_{a_3} Q_{\nu j} I(P_{k_4}\psi_4, P_{k_5}\psi_5))]$$

and distinguish between the following situations:

(1): $k_2 \leq a_1 + O(1)$. We observe the inequality

$$\begin{aligned}&||\nabla^{-1}P_{a_2}(P_{k_3}\psi_3\nabla^{-1}P_{a_3}Q_{\nu j}I(P_{k_4}\psi_4, P_{k_5}\psi_5)||_{L_t^{\frac{4+}{3}}L_x^{8-}}\\ &\lesssim 2^{\delta(\min\{a_2,a_3,k_3\}-\max\{a_2,a_3,k_3\})}\prod_{i=3,4,5}||P_{k_i}\psi_i||_{S[k_i]}\end{aligned}$$

where $\frac{3}{4+}+\frac{2}{8-}=1$, and $\delta>0$ suitably small. This follows easily from lemma 3.1, lemma 6.2, except when $a_3 = k_3 + O(1) >> a_2$. In this case, observe the decomposition

$$\begin{aligned}&\nabla^{-1}P_{a_2}(P_{k_3}\psi_3\nabla^{-1}P_{a_3}Q_{\nu j}I(P_{k_4}\psi_4, P_{k_5}\psi_5))\\ &=\sum_{c_{1,2}\in C_{k_3,a_2-k_3}}\nabla^{-1}P_{a_2}(P_{c_1}\psi_3\nabla^{-1}P_{c_2}Q_{\nu j}I(P_{k_4}\psi_4, P_{k_5}\psi_5))\end{aligned}$$

whence by (the proof of) lemma 6.7, letting $p>4$,

$$\begin{aligned}&||\nabla^{-1}P_{a_2}(P_{k_3}\psi_3\nabla^{-1}P_{a_3}Q_{\nu j}I(P_{k_4}\psi_4, P_{k_5}\psi_5))||_{L_t^{\frac{p}{3}}L_x^{\frac{2p}{p-3}}}\\ &\lesssim 2^{-a_3}\Big(\sum_{c\in C_{k_3,a_2-k_3}}||P_c\psi_3||^2_{L_t^pL_x^\infty}\Big)^{\frac{1}{2}}||P_{a_3}Q_{\nu j}I(P_{k_4}\psi_4, P_{k_5}\psi_5)||_{L_t^2L_x^2}\\ &\leq C_\epsilon 2^{(a_2-k_3)(\frac{1}{p}-\frac{1}{4+\epsilon})}\prod_{i=3,4,5}||P_{k_i}\psi_i||_{S[k_i]}.\end{aligned}$$

We can choose $\epsilon>0$ arbitrarily small. Armed with this estimate, we now compute

$$\begin{aligned}&||P_0[P_{k_1}\psi_1\nabla^{-1}P_{a_1}(P_{k_2}\psi_2\nabla^{-1}P_{a_2}(P_{k_3}\psi_3\nabla^{-1}P_{a_3}[P_{k_4}\psi_4, P_{k_5}\psi_5]))]||_{L_t^1L_x^2}\\ &\leq C2^{-a_1}||P_{k_1}\psi_1||_{L_t^ML_x^{2+}}||P_{k_2}\psi_2||_{L_t^{4+}L_x^\infty}\\ &\qquad\qquad ||\nabla^{-1}P_{a_2}(P_{k_3}\psi_3\nabla^{-1}P_{a_3}[P_{k_4}\psi_4, P_{k_5}\psi_5])||_{L_t^{\frac{4}{3}+}L_x^{8-}}\\ &\leq C2^{-\delta_1|k_1|}2^{\delta_2(k_2-a_1)}2^{\delta_3(\min\{a_2,a_3,k_3\}-\max\{a_2,a_3,k_3\})}2^{-\frac{|k_4-k_5|}{2}}\prod_{i=1,2,3,4,5}||P_{k_i}\psi_i||_{S[k_i]}\end{aligned}$$

for suitable $\delta_{1,2,3}>0$ and $\frac{1}{M}+\frac{3}{4+}+\frac{1}{4+}=1$, and one can now sum over $a_1>-10$ etc. to obtain the claim of the lemma.

(2): $k_2 >> a_1$, $a_3 \leq a_1 + O(1)$. This case doesn't offer anything new. One decomposes

$$P_{a_1}(P_{k_2}\psi_2\nabla^{-1}P_{a_2}(P_{k_3}\psi_3\nabla^{-1}P_{a_3}IQ_{\nu j}(P_{k_4}\psi_4,P_{k_5}\psi_5)))$$
$$=\sum_{c_{1,2}\in C_{k_2,a_1-k_2},\,\mathrm{dist}(c_1,-c_2)\lesssim 2^{a_1+O(1)}}P_{a_1}(P_{c_1}\psi_2\nabla^{-1}P_{a_2}(P_{c_2}\psi_3$$
$$\nabla^{-1}P_{a_3}IQ_{\nu j}(P_{k_4}\psi_4,P_{k_5}\psi_5))$$

Then one invokes lemma 6.2 in order to place $\nabla^{-1}P_{a_3}IQ_{\nu j}(P_{k_4}\psi_4,P_{k_5}\psi_5)$ into $L_t^2L_x^\infty$, and also uses lemma 6.6, exactly as before.

(3): $k_2>>a_1$, $a_1<<a_3<<k_2$. This time, proceeding as in the first case, we get the estimate

$$\lesssim 2^{-a_1}||P_{k_1}\psi_1||_{L_t^ML_x^{2+}}(\sum_{c_1\in C_{k_2,a_1-k_2}}||P_{c_1}\psi_2||^2_{L_t^{4+}L_x^\infty})^{\frac12}$$
$$(\sum_{c_2\in C_{k_2,a_1-k_2}}||\nabla^{-1}P_{c_2}(P_{k_3}\psi_3\nabla^{-1}P_{a_3}IQ_{\nu j}(P_{k_4}\psi_4,P_{k_5}\psi_5))||^2)^{\frac12}$$

Now we have

$$(\sum_{c_2\in C_{k_2,a_1-k_2}}||\nabla^{-1}P_{c_2}(P_{k_3}\psi_3\nabla^{-1}IQ_{\nu j}(P_{k_4}\psi_4,P_{k_5}\psi_5))||^2_{L_t^{\frac{2p}{2+p}}L_x^2})^{\frac12}$$
$$\lesssim(\sum_{c_3\in C_{k_3,a_3-k_3}}\sum_{c_2\in C_{k_2,a_1-k_2},c_2\subset 10c_3}||\nabla^{-1}P_{c_2}(P_{c_3}\psi_3\nabla^{-1}IQ_{\nu j}(P_{k_4}\psi_4,P_{k_5}\psi_5))||^2_{L_t^{\frac{2p}{2+p}}L_x^2})^{\frac12}$$
$$\lesssim(\sum_{c_3\in C_{k_3,a_3-k_3}}||\nabla^{-1}(P_{c_3}\psi_3\nabla^{-1}IQ_{\nu j}(P_{k_4}\psi_4,P_{k_5}\psi_5)||^2_{L_t^{\frac{2p}{2+p}}L_x^2})^{\frac12}$$
$$\lesssim C_\epsilon 2^{-\frac{k_3}{p}}2^{-\frac{a_3}{2}}2^{(\frac{3}{4+}-\frac{2}{p})(a_3-k_2)}2^{-\frac{|k_4-k_5|}{2}}\prod_{i=3,4,5}||P_{k_i}\psi_i||_{S[k_i]}$$

Reiterating application of lemma 6.7, we estimate the expression by (using $4+=p$)

$$\lesssim C_{\epsilon,\delta}2^{-a_1}2^{(\frac{3}{4+\epsilon}-\frac{2}{p})a_1}2^{(\frac{1}{4+\epsilon}-\frac{2}{p})a_3}2^{(\frac{2}{p}-\frac12)k_2}2^{-\frac{|k_4-k_5|}{2}}\prod_{i=1}^5||P_{k_i}\psi_i||_{S[k_i]}$$

One can now sum over the appropriate frequency ranges.

(4): The remaining case $k_2>>a_1$, $a_3\geq k_2+O(1)$ is a monotonous reiteration of the same kind of argument, hence left for the interested reader. ■

Thanks to the preceding lemma, we see that it suffices to consider the following type of (schematically written) expression

$$\sum_{k<-10}\nabla_{x,t}P_0[P_{k_1}\psi_1\nabla^{-1}P_k(P_{k_2}R_\beta I\psi_2\nabla^{-1}(P_{k_3}\psi_3\nabla^{-1}Q_{\nu j}I(P_{k_4}\psi_4,P_{k_5}\psi_5))].$$

As explained in the discussion in section 5.2, we are forced to invoke further dynamic separations here. For example, substituting the gradient components into the first expression on the right-hand side of (8) and including the operators I as before, one obtains the following complicated expression:

$$\begin{aligned}
&N(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5)=\\
&\partial^\nu[\psi_1\sum_j\triangle^{-1}\partial_j(R_jI\psi_2(\sum_i\triangle^{-1}\partial_i[R_\nu\psi_3\sum_k\triangle^{-1}\partial_kI(R_i\psi_4R_k\psi_5-R_i\psi_5R_k\psi_4)]\\
&-\sum_i\triangle^{-1}\partial_i[R_i\psi^3\sum_k\triangle^{-1}\partial_kI(R_\nu\psi_4R_k\psi_5-R_\nu\psi_5R_k\psi_4)])))]\\
&-\partial^\nu[\psi_1\sum_j\triangle^{-1}\partial_j(R_\nu I\psi_2(\sum_i\triangle^{-1}\partial_i[R_j\psi_3\sum_k\triangle^{-1}\partial_kI(R_i\psi_4R_k\psi_5-R_i\psi_5R_k\psi_4)]\\
&-\sum_i\triangle^{-1}\partial_i[R_i\psi_3\sum_k\triangle^{-1}\partial_kI(R_j\psi_4R_k\psi_5-R_j\psi_5R_k\psi_4)])))].
\end{aligned}$$

Similar expressions arise upon enacting complete dynamic separation in the remaining terms in (8). Now we can state:

Proposition 7.2. *Let $N(\psi_1,\dots,\psi_5)$ be one of the quintilinear null-forms described above. Also, assume $||P_k\psi_i||_{S[k]}\leq c_k$ for a 'sufficiently flat*[33] *frequency envelope. Then the following inequality holds:*

$$||P_0N(\psi_1,\dots,\psi_5)||_{N[0]}\lesssim c_0$$

Remark: We note that we no longer require cancellations between the different quintilinear null-forms. This renders the treatment of these somewhat simpler than the trilinear ones. Unfortunately, the size of these expressions makes them somewhat unwieldy.

Proof : We state the proof for the null-form written in detail above. The other null-forms require no separate considerations, as will become clear from the proof. We commence by reducing ψ_3 to hyperbolic microsupport. This is routine now, on account of lemma 6.2, lemma 6.3, similar to the calculations in the Appendix. Similarly, we easily reduce the output to modulation $<O(1)$, so the operator ∂^ν is certainly not going to hurt us. On account of lemma 7.1, we may assume that the expressions $\sum_j\triangle^{-1}(...)$ are microlocalized to frequency $<2^{-10}$. We shall freeze the frequency of these expressions to dyadic size $\sim 2^k$, $k<-10$, and later have to obtain an exponential gain in $-|k-k_i|$ for some i in order to be able to sum. With k fixed, we may also reduce the output, first input and therefore also $\sum_j\triangle^{-1}(...)$ to modulation $<2^{k+O(1)}$, as follows from lemma 6.2 and a straightforward exercise involving lots of applications of Bernstein's inequality. We shall assume these

[33] I. e. the σ used to define it is sufficiently small.

reductions from now on. In order simplify life further, we use the following identity:

$$N(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5)=A_1+A_2+A_3,$$

where we have used the following terminology(keep our reductions in mind):

$$\begin{aligned}A_1 &= \Box[\psi_1\sum_j \triangle^{-1}\partial_j(R_jI\psi_2\sum_i\triangle^{-1}\partial_i[\nabla^{-1}\psi_3\\ &\qquad\qquad \sum_k\triangle^{-1}\partial_k I(R_i\psi_4R_k\psi_5-R_k\psi_4R_i\psi_5)])]\\ A_2 &= -\partial^\nu[\partial_\nu\psi_1\sum_j\triangle^{-1}\partial_j(R_jI\psi_2\\ &\qquad \sum_i\triangle^{-1}\partial_i[\nabla^{-1}\psi_3\sum_k\triangle^{-1}\partial_kI(R_i\psi_4R_k\psi_5-R_k\psi_4R_i\psi_5)])]\\ A_3 &=\\ &-\partial^\nu[\psi_1\sum_j\triangle^{-1}\partial_j[R_jI\psi_2\nabla^{-1}\psi_3\sum_k\triangle^{-1}\partial_kI(R_\nu\psi_4R_k\psi_5-R_k\psi_4R_\nu\psi_5)]]\\ &-\partial^\nu[\psi_1\sum_j\triangle^{-1}\partial_j(R_jI\psi_2\sum_i\triangle^{-1}\partial_i[\nabla^{-1}\psi_3I(R_i\psi_4R_\nu\psi_5-R_\nu\psi_4R_i\psi_5)])]\\ &-\partial^\nu[\psi_1R_\nu I\psi_2\sum_i\partial_i\triangle^{-1}[\nabla^{-1}\psi_3\sum_k\triangle^{-1}\partial_kI(R_i\psi_4R_k\psi_5-R_k\psi_4R_i\psi_5)]]\\ &+\partial^\nu[\psi_1\sum_j\triangle^{-1}\partial_j[R_\nu I\psi_2\nabla^{-1}\psi_3\sum_k\triangle^{-1}\partial_kI(R_j\psi_4R_k\psi_5-R_k\psi_4R_j\psi_5)]]\\ &+\partial^\nu[\psi_1\sum_j\triangle^{-1}\partial_j[R_\nu I\psi_2\sum_i\triangle^{-1}\partial_i[\nabla^{-1}\psi_3I(R_i\psi_4R_j\psi_5-R_j\psi_4R_i\psi_5)]]]\end{aligned}$$

If we frequency-localize the inputs ψ_i to frequency 2^{k_i}, the above identity is useful provided $k_3\geq k_2+O(1)$, which we assume in the following. In the opposite case, it is obvious that one may always let one operator ∇^{-1} fall on $P_{k_2}\psi_2$, and the resulting terms can be treated by minor variations of the following arguments, as explained in the Appendix. We now apply the reductions discussed in the preceding paragraph (whence $k_1=O(1)$). Then $A_{1,2}$ are easy to estimate: Indeed, we can estimate

$$\begin{aligned}&||P_0Q_{<k}\Box[P_{k_1}Q_{<k}\psi_1\sum_j\triangle^{-1}\partial_jP_kQ_{<k+O(1)}(R_jP_{k_2}\psi_2\sum_i\triangle^{-1}\partial_i[\nabla^{-1}P_{k_3}\psi_3\\ &\qquad\qquad\sum_k\triangle^{-1}\partial_kI(R_jP_{k_4}\psi_4R_kP_{k_5}\psi_5-R_kP_{k_4}\psi_4R_jP_{k_5}\psi_5)])]||_{N[0]}\\ &\lesssim 2^{-\frac{k}{2}}||P_{k_1}Q_{<k}\psi_1||_{L_t^\infty L_x^2}||||P_kQ_{<k+O(1)}(R_jP_{k_2}\psi_2\sum_i\triangle^{-1}\partial_i[\nabla^{-1}P_{k_3}\psi_3\\ &\qquad\qquad\sum_k\triangle^{-1}\partial_kI(R_jP_{k_4}\psi_4R_kP_{k_5}\psi_5-R_kP_{k_4}\psi_4R_jP_{k_5}\psi_5)])||_{L_t^2L_x^\infty}\end{aligned}$$

The latter expression can be estimated for example by lemma 6.2 and the Sobolev inequality, which results in the following bound:

$$2^{-\frac{|k-k_3|}{2}}2^{\frac{k_2-k_3}{2}}2^{-\frac{|k_4-k_5|}{2}}\prod_{i=1}^{5}||P_{k_i}\psi_i||_{S[k_i]}.$$

The term A_2 is of course handled similarly. The next term in ascending order of difficulty is the third summand of A_3: when the outer derivative ∂^ν falls on the first input $P_{k_1}\psi_1$, one expands the resulting Q_0 null-form $(\partial^\nu P_{k_1}\psi_1 R_\nu P_{k_2}\psi_2)$. This gives a sum of expressions two factors of which may be placed in $L_t^2L_x^2$, using lemma 6.2, lemma 6.4. These contributions may then be placed into $L_t^1L_x^2$. If, on the other hand, the outer derivative falls on $(R_\nu P_{k_2}\psi_2...)$, one can use non-sharp Strichartz type norms, as in the proof of lemma 7.1. We now begin with the first hard term, the first summand of A_3:

(1): *First summand of* A_3. We note that upon using the identity $R_\nu uR_kv - R_kuR_\nu v = \partial_\nu(\nabla^{-1}uR_kv) - \partial_k(\nabla^{-1}uR_\nu v)$, we can rewrite this term as a sum of two expressions quite similar to the first summand of (25), in which the inputs $\psi_{2,3}$ have been replaced by two bilinear or a trilinear and a linear expression, respectively. It turns out that this helps for estimates. As for the trilinear estimates, we worry mostly about destructive resonances:

(1.1): *High-high interaction between* $(P_{k_2}R_j\psi_2P_{k_3}\nabla^{-1}\psi_3)$ *and* $\sum_k\triangle^{-1}(...)$. Localize the frequency of these terms to dyadic size 2^r, $2^{r+O(1)}$, respectively, where $r>>k$, k as in the preceding discussion. Our first goal shall be to obtain an exponential gain in the difference $\min\{k-\min\{r,\min\{k_2,k_4,k_5\}\},0\}$. Note that we required a special cancellation to achieve the analogous step for the trilinear terms.

(1.1.a): *Obtain an exponential gain in* $\min\{k-\min\{r,\min\{k_2,k_4,k_5\}\},0\}$. We first reduce
$(P_{k_2}R_j\psi_2P_{k_3}\nabla^{-1}\psi_3)$ etc. to modulation $<2^{k+C}$: observe that we have

$$\begin{aligned}&||\partial^\nu P_0Q_{<k}[P_{k_1}Q_{<k}\psi_1\sum_j\triangle^{-1}\partial_jP_kQ_{<k+O(1)}[P_rQ_{\geq k+C}(R_j\psi_2\nabla^{-1}P_{k_3}\psi_3)\\&\qquad\sum_k\triangle^{-1}\partial_kP_{r+O(1)}I(R_\nu P_{k_4}\psi_4R_kP_{k_5}\psi_5-R_kP_{k_4}\psi_4R_\nu P_{k_5}\psi_5)]]||_{N[0]}\\&\lesssim 2^{k-r}||P_{k_1}Q_{<k}\psi_1||_{L_t^\infty L_x^2}||P_rQ_{\geq k+C}(R_jP_{k_2}\psi_2\nabla^{-1}P_{k_3}\psi_3)||_{L_t^2L_x^2}\\&\qquad||P_{r+O(1)}I(R_\nu P_{k_4}\psi_4R_kP_{k_5}\psi_5-R_kP_{k_4}\psi_4R_\nu P_{k_5}\psi_5)||_{L_t^2L_x^2},\end{aligned}$$

where we have used Bernstein's inequality. Now one uses lemma 6.2, lemma 6.4, obtaining an exponential gain $2^{\frac{1}{2+}(k-r)}$. Note that if $(\nabla^{-1}P_{k_2}\psi_2P_{k_3}\psi_3)$ is reduced to modulation $<2^{k+C}$, this forces a similar condition on $I(R_\nu P_{k_4}\psi_4...)$. Thus we need to estimate the following term

$$\begin{aligned}&\partial^\nu P_0Q_{<k}[P_{k_1}Q_{<k}\psi_1\sum_j\triangle^{-1}\partial_jP_kQ_{<k+O(1)}[P_rQ_{<k+C}(R_j\psi_2\nabla^{-1}P_{k_3}\psi_3)\\&\qquad\sum_k\triangle^{-1}\partial_kP_{r+O(1)}Q_{<k+C}I(R_\nu P_{k_4}\psi_4R_kP_{k_5}\psi_5-R_kP_{k_4}\psi_4R_\nu P_{k_5}\psi_5)]],\end{aligned}$$

which we rewrite as the difference of the following two terms:

$$A = \partial^\nu P_0 Q_{<k}[P_{k_1} Q_{<k}\psi_1 \sum_j \triangle^{-1}\partial_j P_k Q_{<k+O(1)}[P_r Q_{<k+C}(R_j\psi_2\nabla^{-1}P_{k_3}\psi_3)$$
$$\sum_k \triangle^{-1}\partial_k P_{r+O(1)} I\partial^\nu(\nabla^{-1}P_{k_4}\psi_4 R_k P_{k_5}\psi_5)]],$$

$$B = \partial^\nu P_0 Q_{<k}[P_{k_1} Q_{<k}\psi_1 \sum_j \triangle^{-1}\partial_j P_k Q_{<k+O(1)}[P_r Q_{<k+C}(R_j\psi_2\nabla^{-1}P_{k_3}\psi_3)$$
$$P_{r+O(1)} I(\nabla^{-1}P_{k_4}\psi_4 R_\nu P_{k_5}\psi_5)]],$$

where we assume w. l. o. g. that $k_4 \geq k_5$. Consider the first expression. Letting the outer derivative fall on the first input (as we may from previous considerations), putting $R_j P_{k_2}\psi_2\nabla^{-1}P_{k_3}\psi_3 = N$, $\sum_k \triangle^{-1}\partial_k(...) = M$, using simple geometric observations as before and expanding the Q_0 null-structure, we schematically reformulate this term as follows:

$$\text{"}2^{-k}\text{"} \sum_{c_{1,2}\in C_{r,k-r},\, \mathrm{dist}(c_1,-c_2)\lesssim 2^k} P_0 Q_{<k}[\Box(P_{k_1}Q_{<k}\psi_1 P_{c_1}Q_{<k+C}M)P_{c_2}Q_{<k+C}N]$$
$$- 2^{-k} \sum_{c_{1,2}\in C_{r,k-r},\, \mathrm{dist}(c_1,-c_2)\lesssim 2^k} P_0 Q_{<k}[\Box P_{k_1}Q_{<k}\psi_1 P_{c_1}Q_{<k+C}M P_{c_2}Q_{<k+C}N]$$
$$- 2^{-k} \sum_{c_{1,2}\in C_{r,k-r},\, \mathrm{dist}(c_1,-c_2)\lesssim 2^k} P_0 Q_{<k}[P_{k_1}Q_{<k}\psi_1 P_{c_1}\Box Q_{<k+C}M P_{c_2}Q_{<k+C}N].$$

We have as usual replaced the operator $P_k Q_{<k+O(1)}\nabla^{-1}$ by "2^{-k}", and keep in mind that making things rigorous would involve writing everything out using convolution kernels. The preceding three terms are straightforward to estimate: for example, we have

$$||\text{"}2^{-k}\text{"} \sum_{c_{1,2}\in C_{r,k-r},\, \mathrm{dist}(c_1,-c_2)\lesssim 2^k} P_0 Q_{<k}[\Box(P_{k_1}Q_{<k}\psi_1 P_{c_1}Q_{<k+C}M)$$
$$P_{c_2}Q_{<k+C}N]||_{N[0]}$$
$$\lesssim 2^{-k} \sum_{c_{1,2}\in C_{r,k-r},\, \mathrm{dist}(c_1,-c_2)\lesssim 2^k} 2^{\min\{r,0\}}||\Box(P_{k_1}Q_{<k}\psi_1 P_{c_1}Q_{<k+C}M)||_{\dot{X}^{0,-\frac{1}{2},1}_{\leq\max\{r,0\}}}$$
$$2^{\frac{k-\min\{r,0\}}{2+}}||P_{c_2}Q_{<k+C}N||_{\dot{X}^{0,\frac{1}{2},1}_r},$$

where we have invoked a simple modification[34] of lemma 6.5 as well as (19). If one now freezes the modulation of M, N to dyadic values, applies Cauchy-Schwartz to get rid of the summation over discs and then Plancherel's theorem, then applies lemma 6.4 three times and sums over the dyadic modulations, one obtains the following bound:

[34]We use that $||P_0[P_{k_1}\psi P_c F]||_{N[0]} \lesssim 2^{\frac{k-\min\{r,0\}}{2+}} 2^{\min\{r,0\}}||P_{k_1}\psi||_{\dot{X}^{0,\frac{1}{2},1}_{k_1}}||P_c F||_{\dot{X}^{0,-\frac{1}{2},1}_r}$ provided $c \in C_{r,k-r}$.

$$C_\epsilon 2^{\min\{r,0\}-k}2^{(k-\min\{r,0\})(1-\epsilon)}2^{k_2-k_3}2^{k_5-k_4}$$
$$2^{\frac{\min\{k-\min\{r,k_2\},0\}}{4+}}2^{\frac{\min\{k-\min\{r,k_5\},0\}}{4+}}\prod_{i=1}^{5}||P_{k_i}\psi_i||_{S[k_i]}.$$

Since $\epsilon > 0$ here is arbitrary, this yields the desired estimate. The remaining cases above are easier and left for the reader (one can place both M, N into $L_t^4 L_x^\infty$, using the improved Strichartz estimate of Klainerman-Tataru, see lemma 6.7.) The term B from before is of course treated similarly: distinguish between $k_5 \leq k+O(1)$ and $k_5 >> k$. In the former case, one can use theorem 6.9 directly (which produces an exponential gain in $k_5 - k$) in conjunction with the inequality

$$||P_k[P_r Q_{<k+C}(R_j\psi_2\nabla^{-1}P_{k_3}\psi_3)\nabla^{-1}P_{k_4}T_y\psi_4]||_{\dot{X}_k^{0,\frac{1}{2},1}}$$
$$\lesssim 2^{\frac{k-r}{4+}}2^{k_2-k_3}\prod_{i=2,3,4}||P_{k_i}\psi_i||_{S[k_i]},$$

in which T_y denotes the translation operator $f(.) \to f(.-y)$. In the latter case, one proceeds as before, reducing the modulation of $R_\nu P_{k_5}\psi_5$ suitably, microlocalizing to discs, expanding the Q_0 structure and invoking lemma 6.12, which results in gains both in $k-r$ and $r - \max_{i=2,3,4,5}\{k_i\}$. This is even more than we need.
We now observe that provided either $k_2 = r + O(1)$ or $k_5 = r + O(1)$, using lemma 6.4 and the trilinear estimates proved in section 6 allow us to obtain exponential gains in $k_2 - k_3$, $k_5 - k_4$ for term A, while this was already achieved for term B in the above discussion. This in combination with the above allows us to sum over $k, k_2, \dots k_5$ in order to obtain the desired estimate. Thus the only potential problem occurs in case of high-high interactions in both inner curly brackets in A, i. e. $k_2 = k_3 + O(1) >> r$, $k_4 = k_5 + O(1) >> r$.
(1.1.b): *Assuming $k_2 = k_3 + O(1)$, $k_4 = k_5 + O(1)$, obtain exponential gain in $r - \min\{k_i\}$.* In order to achieve this, we shall invoke improved Strichartz type norms as in lemma 7.1. We consider again the expression A but with the operator $Q_{<k}$ applied to $(R_j P_{k_2}\psi_2\nabla^{-1}P_{k_3}\psi_3)$ replaced by $Q_{<\tilde{r}}$ where $\tilde{r} = \min\{r+10, -10\}$ (we can reduce the upper bound for k for that purpose, as this is irrelevant for the validity of lemma 7.1). We next reduce both $P_{k_4}\psi_4$, $P_{k_5}\psi_5$ to modulation $< 2^{\tilde{r}}$. For simplicity's sake, introduce the following quantity:

$$A^{\pm,\pm,\pm}(P_{k_1}\psi_1,\dots,P_{k_5}\psi_5) := \sum_{j,k}\partial^\nu P_{k_1}Q^{\pm}_{<k}\psi^1\triangle^{-1}\partial_j P_k Q_{<k+C}[R_j P_{k_2}\psi_2\nabla^{-1}P_{k_3}\psi_3$$
$$\triangle^{-1}\partial_k\partial_\nu P_r Q_{<\tilde{r}}(\nabla^{-1}P_{k_4}Q^{\pm}_{<\tilde{r}}\psi_4 R_k P_{k_5}Q^{\pm}_{<\tilde{r}}\psi_5)].$$

Proceeding as in (28) and introducing $\tilde{k}_4 = \min\{k_4, 0\}$, we decompose this further as follows:

$$A^{\pm,\pm,\pm}(P_{k_1}\psi_1,\dots,P_{k_5}\psi_5) = \tag{30}$$

$$= \sum_{a>0} \sum_{\omega_{1,2,3}\in K_{\frac{\tilde{r}-\tilde{k}_4+a}{2}-10},\, \max\{\mathrm{dist}(\pm\omega_1,\pm\omega_{2,3})\}\sim 2^{\frac{\tilde{r}-\tilde{k}_4+a}{2}}} A^{\pm,\pm,\pm}(P_{k_1,\omega_1}\psi_1,\dots,P_{k_4,\omega_2}\psi_4,P_{k_5,\omega_3}\psi_5)$$
$$+ \sum_{\omega_{1,2,3}\in K_{\frac{\tilde{r}-\tilde{k}_4}{2}-10},\, \mathrm{dist}(\pm\omega_1,\pm\omega_{2,3})\lesssim 2^{\frac{\tilde{r}-\tilde{k}_4}{2}}} A^{\pm,\pm,\pm}(P_{k_1,\omega_1}\psi_1,\dots,P_{k_4,\omega_2}\psi_4,P_{k_5,\omega_3}\psi_5).$$

Consider the first double sum. W. l. o. g. we may assume $\mathrm{dist}(\pm\omega_1,\pm\omega_2)\sim 2^{\frac{\tilde{r}-\tilde{k}_4+a}{2}}$. We note that we may concurrently microlocalize $P_{k_4,\omega_1}\psi_4$, $P_{k_5,\omega_3}\psi_5$ to discs $c_{1,2}\in C_{k_{4,5},r-k_{4,5}}$ of radius 2^r with the property $\mathrm{dist}(c_1,-c_2)\lesssim 2^r$. Assume first that $k_4<-10$. Then we have the identity

$$\begin{aligned}&\partial^\nu P_{k_1,\omega_1}Q^{\pm}_{<k}\psi_1\nabla^{-1}P_{k_4,\omega_2}Q^{\pm}_{<\tilde{r}}\psi_4\\ &=P_{O(1)}Q_{\geq\tilde{r}+a+O(1)}[\partial^\nu P_{k_1,\omega_1}Q^{\pm}_{<k}\psi_1\nabla^{-1}P_{k_4,\omega_2}Q^{\pm}_{<\tilde{r}}\psi_4].\end{aligned}$$

If we now rearrange terms and commit abuse of notation, we can estimate

$$\begin{aligned}&||P_0A^{\pm,\pm,\pm}(P_{k_1,\omega_1}\psi_1,\dots,P_{k_4,\omega_2}\psi_4,P_{k_5,\omega_3}\psi_5)||_{N[0]}\\ &\lesssim 2^{-k}\sum_{c_{1,2}\in C_{k_{4,5}},\mathrm{dist}(c_1,-c_2)\lesssim 2^r}||\partial^\nu P_{k_1,\omega_1}Q^{\pm}_{<k}\psi_1\nabla^{-1}P_{k_4,\omega_2}Q^{\pm}_{<\tilde{r}}P_{c_1}\psi_4||_{L^2_tL^2_x}\\ &\qquad ||P_r(P_{k_2}\psi_2\nabla^{-1}P_{k_3}\psi_3)P_{c_2}P_{k_5,\omega_3}Q^{\pm}_{<\tilde{r}}\psi_5||_{L^2_tL^\infty_x}\\ &\lesssim 2^{-k}\sum_{c_{1,2}\in C_{k_{4,5}},\mathrm{dist}(c_1,-c_2)\lesssim 2^r}2^{-\frac{\tilde{r}+a}{2}}||\partial^\nu P_{k_1,\omega_1}Q^{\pm}_{<k}\psi_1\nabla^{-1}P_{k_4,\omega_2}Q^{\pm}_{<\tilde{r}}P_{c_1}\psi_4||_{\dot{X}_0^{0,\frac12,1}}\\ &\qquad ||P_r(P_{k_2}\psi_2\nabla^{-1}P_{k_3}\psi_3)P_{c_2}P_{k_5,\omega_3}Q^{\pm}_{<\tilde{r}}\psi_5||_{L^2_tL^\infty_x},\end{aligned}$$

where we have used lemma 6.4 in the last step. If we now refer to lemma 6.7, we can estimate

$$\begin{aligned}&||P_r(P_{k_2}\psi_2\nabla^{-1}P_{k_3}\psi_3)P_{c_2}P_{k_5,\omega_3}Q^{\pm}_{<\tilde{r}}\psi_5||_{L^2_tL^\infty_x}\\ &\qquad\lesssim 2^{\frac{2r}{3}}2^{\frac{r-k_3}{6+}}\prod_{i=2,3}||P_{k_i}\psi_i||_{S[k_i]}||P_{c_2}Q_{<\tilde{r}}\psi_5||_{L^6_tL^\infty_x}.\end{aligned}$$

We now carry out the summation over $c_{1,2}$ (keeping in mind that for fixed $c_{1,2}$ there are only finitely many poosibilities for $\omega_{1,2}$) and apply Cauchy-Schwarz as well as lemmata 6.4[35], 6.7, to obtain the upper bound

$$\begin{aligned}&\lesssim 2^{-k}2^{-\frac{\tilde{r}+a}{2}}2^{\frac{r-k_4}{2+}}2^{\frac{5k_5}{6}}2^{\frac{5(r-k_5)}{12+}}2^{\frac{2r}{3}}2^{\frac{r-k_3}{6+}}\prod_{i=1}^5||P_{k_i}\psi_i||_{S[k_i]}\\ &\lesssim 2^{r-k}2^{\frac{r-k_5}{12+}}2^{\frac{r-k_3}{6+}}\prod_{i=1}^5||P_{k_i}\psi_i||_{S[k_i]}.\end{aligned}$$

[35]More precisely, we apply a simple modification mentioned before.

If one now takes a suitable geometric means with the estimate obtained in section **(1.1.a)**, one gets exponential gains both in $k-r$ as well as $r-\min\{k_i\}$, as desired. The cases $k_4\in[-10,10]$, $k_4>10$ are handled completely analogously to the discussion following (28) as well as to the above, and hence left out. As to the 2nd sum in (30), we discard the operator $\sum_k \triangle^{-1}\partial_k P_r Q_{<\tilde{r}}$, keeping in mind that this costs 2^{-r}, and let the derivative ∂_ν fall on either $P_{k_4,\omega_1}\psi_4$ or $P_{k_5,\omega_3}\psi_5$. Then one expands the resulting Q_0-structure, and argues exactly as before. The exponential gains coming from the null-form expansion make up for the 2^{-r}-loss, and one proceeds using lemma 6.7, exactly as above. This concludes step **(1.1.b)** and thereby step **(1.1)**.

(1.2): *Low-high/high-low interaction between* $(P_{k_2}R_j\psi_2 P_{k_3}\nabla^{-1}\psi_3)$ *and* $\sum_k \triangle^{-1}(...)$. Freezing the frequencies of these expressions to dyadic size $\sim 2^{r_{1,2}}$ respectively, one gets an exponential gain in $-|r_1-r_2|$ using theorem 6.9. If there is no high-high interaction within one of these terms, using lemma 6.4 one also has $r_1=k_3+O(1)$ or $r_2=k_4+O(1)$, and obtains an exponential gain in $-|k_2-k_3|$ or $-|k_4-k_5|$, respectively. This is enough to sum over all localization parameters. If there are only high-high interactions within these expressions, one proceeds exactly as in the previous number. The details are tedious reiterations.

(2): *Second summand of* A_3. We begin with the case when there is a high-high interaction between $R_jI\psi_2$ and $\sum_i \triangle^{-1}\partial_i[...]$. We schematically represent the corresponding term as

$$\partial^\nu[\psi_1\nabla^{-1}[\nabla^{-1}\psi_2\nabla^{-1}\psi_3 I Q_{\nu j}(\psi_4,\psi_5)]]$$

and keep our assumptions about frequencies and modulations in mind. We proceed as in **(1)** for the first term of A_3, splitting this into two terms, the first of which is treated exactly as in **(1)**; indeed, one checks that one gains $2^{\frac{5(k_2-k_3)}{4}}$ there, so we may lose $2^{k_3-k_2}$. We now consider the 2nd term, which has the schematic form

$$\partial^\nu\psi_1\nabla^{-1}P_k[\nabla^{-1}\psi_2\nabla^{-1}\psi_3\psi_4 R_\nu\psi_5].$$

We need to pair off the k_i, $i=1,2,3,4$ and gain exponentially in the differences. Our assumptions are $k_2<k_3+O(1)$, $k<<k_2$. We deal with several cases:

(2.1): $k_5\leq k+O(1)$. Reduce ψ_5 and ψ_1 to modulation $<2^{k_5}$. Then expand the Q_0-null-structure and place $\nabla^{-1}\psi_2\nabla^{-1}\psi_3\psi_4$ into $L_t^2L_x^{2+}$ using the Strichartz $L_t^6L_x^{6+}$. One gains exponentially in k_5-k_2, k_4-k_3.

(2.2): $k_2\geq k_5>>k$. Carrying out the analysis of the preceding number yields an exponential gain in k_5-k_2, k_4-k_3. We need also an exponential gain in $k-k_5$. Use the inequality

$$||P_{k_5}[\nabla^{-1}P_{k_2}\psi_2\nabla^{-1}P_{k_3}\psi_3P_{k_4}\psi_4]||_{\dot{X}_{k_5}^{0,\epsilon,1}}\lesssim 2^{-\frac{k_5}{4+}-\frac{k_2}{4}}\prod_i ||P_{k_i}\psi_i||_{S[k_i]}.$$

This allows easily to reduce the modulations of $R_\nu\psi_5$ as well as $\nabla^{-1}\psi_2\nabla^{-1}\psi_3\psi_4$ to size $<2^k$. Then one rewrites the expression as

$$\sum_{c_i\in C_{k_5,k-k_5},\,\mathrm{dist}(c_1,-c_2)\lesssim 2^k} "2^{-k}"\partial^{\nu}\psi_1 R_{\nu}P_{c_1}\psi_5 P_{c_2}[\nabla^{-1}\psi_2\nabla^{-1}\psi_3\psi_4],$$

again uses the above trilinear algebra estimate in conjunction with the customary versions of lemma 6.5 used for the trilinear estimates. One gets the desired exponential gain in $k-k_5$, without loss in the other differences.
(2.3): $k_5 > k_2, k_5 >> k$. One argues as in the preceding number. The gain in $k-k_5$ implies a gain in k_2-k_5.
Now we analyze the case when there is a high-low interaction between $R_j I\psi_2$ and $\sum_i \triangle^{-1}\partial_i[...]$, the low-high case being similar. We represent this schematically as follows:

$$\partial^{\nu}\psi_1\nabla^{-1}\psi_2\nabla^{-1}[\nabla^{-1}\psi_3 IQ_{\nu j}(\psi_4,\psi_5)],$$

which up to terms estimable by the technique of lemma 7.1 is equivalent to

$$\partial^{\nu}(\psi_1\nabla^{-1}P_{k_2}\psi_2)\nabla^{-1}P_k[\nabla^{-1}\psi_3 IQ_{\nu j}(\psi_4,\psi_5)],\ k_2\geq k+O(1).$$

(2.4): $k_5 < k+O(1)$. Reduce $\partial^{\nu}(\psi_1\nabla^{-1}\psi_2)$ to modulation $< 2^{k_5}$. This is achieved by using

$$||\nabla^{-1}P_{k_3}\psi_3 P_{k_4}\psi_4 R_{\nu}P_{k_5}\psi_5||_{L_t^2L_x^{2+}}\lesssim 2^{\frac{k_5+k_3+k_4}{2-}-k_3}\prod_{i=3,4,5}||P_{k_i}\psi_i||_{S[k_i]}$$

in conjunction with lemma 6.4. Similarly, reduce $R_{\nu}\psi_5$ to hyperbolic microsupport. Then expand the null-structure, thereby gaining exponentially in k_5-k_2, k_4-k_3.
(2.5): $k_2\geq k_5 >> k$. Now one also needs to gain exponentially in $k-k_5$. Reduce $R_{\nu}\psi_5$ and $(\nabla^{-}\psi_3\psi_4)$ to modulation $< 2^k$. Then represent the term schematically as

$$\sum_{c_i\in C_{k_5,k-k_5},\,\mathrm{dist}(c_1,-c_2)\lesssim 2^k} "2^{-k}"\partial^{\nu}(\psi_1\nabla^{-1}\psi_2)R_{\nu}P_{c_1}\psi_5 P_{c_2}(\nabla^{-1}\psi_3\psi_4).$$

One estimates it as in **(2.3)**.
(2.6): $k_5\geq k_2$, $k_5 >> k$. This is dealt with as in the preceding number.
(3): *The fourth and fifth summand of* A_3. Our only potential enemy is a high-high between $R_{\nu}\psi_2$ and $\nabla^{-1}\psi_3\sum_k(...)$ or $\sum_i\triangle^{-1}\partial_i(...)$, respectively, since otherwise theorem 6.9 in conjunction with lemma 6.2 settles this case. We focus on the fourth term, the fifth being treated analogously. We may let ∂^{ν} fall on the high-frequency first input, and freeze $\sum_j\triangle^{-1}\partial_j(...)$ to frequency $\sim 2^k$, $k<-10$ (lemma 7.1). Then we freeze the remaining frequencies of ψ_i to 2^{k_i}, letting $k_2 >> k$. Now we concurrently microlocalize the spatial Fourier support of $R_{\nu}P_{k_2}\psi_2$ and $\nabla^{-1}P_{k_3}\psi_3...$ to discs $c_{1,2}\in C_{k_2,k-k_2}$, where $\mathrm{dist}(c_1,-c_2)\lesssim 2^k$, as usual. Fixing such a pair of discs, replacing the operator $P_k\nabla^{-1}$ by "2^{-k}, we claim that we can reduce $(\partial^{\nu}P_{k_1}\psi_1 R_{\nu}P_{c_2}\psi_2)$ to modulation $< 2^{2k-k_2}$. This follows from lemma 6.2, lemma 6.4 as before. One can sum over $c_{1,2}$ using Cauchy-Schwarz, Plancherel,

and a simple modification of lemma 6.4. Further, we may easily reduce $P_{k_{1,2}}\psi_{1,2}$ to modulation $< 2^k$. Then we expand the null-structure. Using lemma 6.5 as in footnote 25, one gains $2^{\frac{k}{2}}2^{\frac{k-\min\{k_2,0\}}{2+}}$, which counteracts the 2^{-k}-loss. This finishes the treatment of the quintilinear null-form $N(\psi_1, \psi_2, \dots \psi_5)$. ■

We now proceed to the first kind of quintilinear term discussed in section 5.2. Recall that it has the schematic form

$$\nabla_{x,t}[\nabla^{-1}(\psi\nabla^{-1}(\psi^2))\nabla^{-1}IQ_{\nu j}(\psi,\psi)].$$

In order to treat it as well as all the remaining error terms, we shall need the following

Lemma 7.3. *Let ψ_ν, $\nu = 0, 1, 2$ and χ_ν be as in section 2. Then provided*

$$||P_k\psi_\nu||_{S[k]([-T,T]\times\mathbf{R}^2)} \leq c_k$$

for a 'suitably flat' frequency envelope[36]*, we have*

$$||P_k\chi_\nu||_{L_t^2L_x^2([-T,T]\times\mathbf{R}^2)} \lesssim 2^{-\frac{k}{2}}c_k.$$

Proof : We use schematic notation. Enacting dynamic separation, we represent $\chi_\nu|_{[-T,T]}$ as follows:

$$\nabla^{-1}[\psi\nabla^{-1}Q_{\nu j}(\psi,\psi)] + \nabla^{-1}[\psi\nabla^{-1}(\nabla^{-1}[\psi\nabla^{-1}(\psi^2)]\psi)] \\ + \nabla^{-1}(\psi\nabla^{-1}(\nabla^{-1}[\psi\nabla^{-1}(\psi^2)]\nabla^{-1}[\psi\nabla^{-1}(\psi^2)]))$$

Treating the first summand is routine after the calculations in theorem 6.10 as well as lemma 6.2. We proceed to the 2nd summand. We may localize it to frequency ~ 1, using scale invariance. We frequency-localize as follows:

$$P_0\nabla^{-1}[P_{k_1}\psi_1\nabla^{-1}P_{a_1}(\nabla^{-1}P_{a_2}[P_{k_2}\psi_2\nabla^{-1}P_{a_3}(P_{k_3}\psi_3P_{k_4}\psi_4)]P_{k_5}\psi_5)].$$

We treat several cases, and of course substitute appropriate Schwartz functions for the (frequency localized) inputs:

(1.1): $k_1 > 10$, $k_5 > a_1 + 10$, $a_3 < k_2 - 10$. Note that under these conditions $a_1 = k_1 + O(1)$, $k_5 = a_2 + O(1) = k_2 + O(1)$. Further assume $a_3 >> a_1$, the other case $a_3 \leq a_1 + O(1)$ being treated more or less identically. Observe that we have

[36]We assume the Wave Map exists on $[-T,T]\times\mathbf{R}^2$.

the identity

$$P_{a_1}(\nabla^{-1}P_{a_2}[P_{k_2}\psi_2\nabla^{-1}P_{a_3}(P_{k_3}\psi_3 P_{k_4}\psi_4)]P_{k_5}\psi_5)$$
$$= \sum_{c_{1,2}\in C_{k_{2,5},a_3-k_{2,5}},\,\mathrm{dist}(c_1,-c_2)\lesssim 2^{a_3}} P_{a_1}(\nabla^{-1}P_{a_2}[P_{c_1}\psi_2\nabla^{-1}P_{a_3}(P_{k_3}\psi_3 P_{k_4}\psi_4)]P_{c_2}\psi_5)$$

Now use lemma 6.7. This yields the estimate for $p > 4$ a small perturbation of 4:

$$||P_{a_1}(\nabla^{-1}P_{a_2}[P_{k_2}\psi_2\nabla^{-1}P_{a_3}(P_{k_3}\psi_3 P_{k_4}\psi_4)]P_{k_5}\psi_5)||_{L_t^{\frac{p}{2}}L_x^{1+}}$$
$$\leq C_{\epsilon,\delta}2^{(-\frac{1}{2+\epsilon}+\frac{2}{p})k_2}2^{(\frac{1}{2+\delta}-\frac{4}{p})a_3}2^{-\mu(\delta)|k_3-k_4|},$$

where $\epsilon, \delta > 0$ can be chosen independently small. With this, we can estimate

$$||P_0\nabla^{-1}[P_{k_1}\psi_1\nabla^{-1}P_{a_1}(\nabla^{-1}P_{a_2}[P_{k_2}\psi_2\nabla^{-1}P_{a_3}(P_{k_3}\psi_3 P_{k_4}\psi_4)]P_{k_5}\psi_5)]||_{L_t^2L_x^2}$$
$$\lesssim 2^{-k_1}||P_{k_1}\psi_1||_{L_t^M L_x^{2+}}||P_{a_1}(\nabla^{-1}P_{a_2}[P_{k_2}\psi_2\nabla^{-1}P_{a_3}(P_{k_3}\psi_3 P_{k_4}\psi_4)]P_{k_5}\psi_5)||_{L_t^{\frac{p}{2}}L_x^2}$$
$$\lesssim 2^{-\frac{k_1}{2+}}2^{\frac{k_1-a_3}{2+}}2^{\delta_1(a_3-k_2)}2^{-\delta_2|k_3-k_4|}\prod_{i=1}^{5}||P_{k_i}\psi_i||_{S[k_i]},$$

where we let $\frac{1}{M}+\frac{2}{p}=\frac{1}{2}$, $\frac{1}{M}+\frac{1}{2}\frac{1}{2+}<\frac{1}{4}$. One can sum over a_3, and gets the desired exponential gains.

(1.2): $k_1 > 10$, $k_5 > a_1 + 10$, $a_3 \geq k_2 - 10$. One can argue as before even without using improved Strichartz type norms: the two operators $P_{a_{2,3}}\nabla^{-1}$ counteract the exponential losses arising from the use of $L_t^{4+}L_x^{\infty}$.

(1.3): $k_1 > 10$, $k_5 \leq a_1 + 10$. This is similar to the preceding case. One gets by using $L_t^{4+}L_x^{\infty}$, as is easily seen.

(2): The remaining cases $k_1 \in [-10, 10]$, $k_1 < -10$ are treated analogously. One simply places $P_{a_1}\nabla^{-1}(...)$ into $L_t^{2+}L_x^{\infty}$. This concludes treatment of the 2nd summand.
In order to treat the *third summand* in the above expansion of χ_ν, we observe that for arbitrary ϵ, $\delta > 0$ and 'sufficiently flat' frequency envelope,

$$||\nabla^{-1}P_k[\psi\nabla^{-1}(\psi^2)]||_{L_t^{4+\epsilon}L_x^{2+\delta}} \lesssim 2^{(\frac{\delta}{2+\delta}-\frac{1}{4+\epsilon})k}c_k.$$

Estimation of the third summand is now an easy exercise, involving simple frequency trichotomies left for the interested reader. ■

It is now straightforward to estimate the first type of quintilinear term: Note that by lemma 6.3, lemma 7.3, denoting the atomic Banach space whose atoms consist of functions ψ satisfying $||\psi||_{L_t^2 L_x^2} \leq 1$, $||\psi||_{L_t^2 L_x^\infty} \leq 1$ and similarly the space B with the preceding Lebesgue spaces replaced by $L_t^\infty L_x^2$, $L_t^\infty L_x^\infty$,

$$\begin{aligned}&||P_0 Q_{>0}[\nabla^{-1}(\psi\nabla^{-1}(\psi^2))\nabla^{-1} I Q_{\nu j}(\psi^2)]||_{L_t^2 L_x^2}\\ &\lesssim ||[\nabla^{-1}(\psi\nabla^{-1}(\psi^2))]||_A ||\nabla^{-1} I Q_{\nu j}(\psi,\psi)||_B \lesssim c_0\epsilon^4,\end{aligned}$$

provided the assumptions in Proposition 4.1 are satisfied. We also have

$$||\nabla^{-1} P_k(\psi\nabla^{-1}(\psi^2))||_{L_t^M L_x^2} \lesssim 2^{-\frac{k}{M}} c_k$$

for M as in the definition of $S[k]$, whence lemma 6.3 implies

$$||P_0 Q_{>0}[\nabla^{-1}(\psi\nabla^{-1}(\psi^2))\nabla^{-1} I Q_{\nu j}(\psi^2)]||_{L_t^M L_x^2} \lesssim c_0\epsilon^4.$$

Next, using lemma 6.2, we estimate

$$||P_0 Q_{\leq 0}[\nabla^{-1}(\psi\nabla^{-1}(\psi^2))\nabla^{-1} I Q_{\nu j}(\psi^2)]||_{L_t^1 L_x^2} \lesssim \epsilon^4 c_0.$$

This concludes the treatment of the quintilinear terms.

8. The remaining error terms.

Careful examination of the manipulations in the preceding sections reveals that the following schematically written error terms have been generated (the last two terms in the list are generated by applying the above described process to the 2nd and third summand of (8)):

$$\nabla_{x,t}[\psi\nabla^{-1}(\chi_\nu\chi_i)]$$

$$\nabla_{x,t}[\psi\nabla^{-1}(R_\beta I\psi\nabla^{-1}(\psi\nabla^{-1}(R_\nu\psi\chi_\mu)))]$$

$$\nabla_{x,t}[\psi\nabla^{-1}(R_\beta I\psi\nabla^{-1}(\psi\nabla^{-1}(\chi_\nu\chi_\mu)))]$$

$$\nabla_{x,t}[\psi\nabla^{-1}(R_\beta I\psi\nabla^{-1}(\chi\nabla^{-1} I Q_{\nu j}(\psi,\psi)))]$$

$$\nabla_{x,t}[\chi\nabla^{-1} I(R_\beta I\psi\nabla^{-1}(\psi\nabla^{-1} I Q_{\nu j}(\psi,\psi)))]$$

$$\nabla_{x,t}[\chi\nabla^{-1} I(R_\beta I\psi\nabla^{-1}(R_\nu I\psi\nabla^{-1} I Q_{\nu j}(\psi,\psi)))]$$

We commence with the first term of the list: as usual, we invoke the following decomposition:

$$\nabla_{x,t}[\psi\nabla^{-1}(\chi_\nu\chi_i)]|_{[-T,T]} = \nabla_{x,t}[\psi\nabla^{-1}((1-I)\chi_\nu(\psi_i - R_i\sum_{k=1,2}R_k\psi_k))] + \nabla_{x,t}[\psi\nabla^{-1}(I\chi_\nu\chi_i)]|_{[-T,T]}.$$

The first summand on the right hand side is treated like the corresponding term in (34), using lemma 7.3, as well as the fact that $||P_k\chi_\nu||_{L_t^M L_x^2} \lesssim 2^{-\frac{k}{M}}c_k$. For the 2nd summand on the right hand side, note that $||P_k I\chi_\nu||_{L_t^\infty L_x^2} \lesssim c_k$. Obviously, one can place the portion of this expression which has Fourier support in the hyperbolic region into $L_t^1 L_x^2$, using the customary Littlewood-Paley rigmarole. The third and fourth term are treated by exact analogy (note that not both μ, ν can be zero). We now turn to the 2nd term in the list: we reduce $R_\nu\psi$ to hyperbolic microsupport as usual and enact one additional dynamic separation in the expression for χ given in (11), replacing it by the sum of terms of the following three schematic forms:

$$\nabla_{x,t}[\psi\nabla^{-1}[R_\beta I\psi\nabla^{-1}(\psi\nabla^{-1}(R_\nu I\psi\nabla^{-1}(\psi\nabla^{-1}Q_{\nu j}(\psi,\psi))))]] \tag{31}$$

$$\nabla_{x,t}[\psi\nabla^{-1}[R_\beta I\psi\nabla^{-1}(\psi\nabla^{-1}(R_\nu I\psi\nabla^{-1}(\psi\nabla^{-1}(R_\nu\psi\chi))))]] \tag{32}$$

$$\nabla_{x,t}[\psi\nabla^{-1}[R_\beta\psi\nabla^{-1}(\psi\nabla^{-1}(R_\nu I\psi\nabla^{-1}(\psi\nabla^{-1}(\chi\chi))))]]. \tag{33}$$

The same comment applies as in the preceding footnote. Arguing as in the preceding, we reduce $Q_{\nu j}(\psi,\psi)$ as well as the output to 'hyperbolic microsupport'. With these reductions in place, we treat the first of the terms above, the others following similarly. We use Strichartz type norms: first observe from lemma 3.1 that

$$||P_{a_1}|\nabla_x|^{-\frac{3+}{8}}[P_{k_1}\psi\nabla^{-1}P_{a2}IQ_{\nu j}(P_{k_2}\psi,P_{k_3}\psi)||_{L_t^{\frac{8}{5}}L_x^{2+}} \lesssim 2^{\delta(\min\{a_1,k_1,a_2\}-\max\{a_1,k_1,a_2\})}2^{-\frac{|k_2-k_3|}{2}}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}.$$

Then we further observe the elementary inequalities

$$||P_{a_1}|\nabla_x|^{-\frac{3+}{8}}(P_{k_1}\psi|\nabla_x|^{-\frac{5-}{8}}P_{a_2}F)||_{L_t^{\frac{4}{3}}L_x^{\frac{16+}{7}}} \lesssim 2^{\delta(\min\{a_1,k_1,a_2\}-\max\{a_1,k_1,a_2\})}||P_{k_1}\psi_1||_{S[k_1]}||P_{a_2}F||_{L_t^{\frac{8}{5}}L_x^{2+}}$$

$$||P_{a_1}|\nabla_x|^{-\frac{3+}{8}}(P_{k_1}\psi|\nabla_x|^{-\frac{5-}{8}}P_{a_2}F)||_{L_t^{\frac{8}{7}}L_x^{\frac{8+}{3}}} \lesssim 2^{\delta(\min\{a_1,k_1,a_2\}-\max\{a_1,k_1,a_2\})}||P_{k_1}\psi_1||_{S[k_1]}||P_{a_2}F||_{L_t^{\frac{8}{5}}L_x^{\frac{16+}{7}}}$$

$$||P_{a_1}|\nabla_x|^{-\frac{3+}{8}}(P_{k_1}\psi|\nabla_x|^{-\frac{5-}{8}}P_{a_2}F)||_{L_t^1L_x^{\frac{16+}{5}}} \lesssim 2^{\delta(\min\{a_1,k_1,a_2\}-\max\{a_1,k_1,a_2\})}||P_{k_1}\psi_1||_{S[k_1]}||P_{a_2}F||_{L_t^{\frac{8}{7}}L_x^{\frac{8+}{3}}}.$$

If we put these estimates together, we conclude that

$$\begin{aligned}&||\nabla_{x,t}P_0 I[P_{k_1}\psi\nabla^{-1}[R_\nu P_{k_2}I\psi\nabla^{-1}(P_{k_3}\psi\\ &\qquad\qquad\nabla^{-1}(R_\nu P_{k_4}I\psi\nabla^{-1}(P_{k_5}\psi\nabla^{-1}IQ_{\nu j}(P_{k_6}\psi,P_{k_7}\psi))))]]||_{L_t^1L_x^2}\\ &\lesssim 2^{-\delta|k_1|}2^{\delta(\min\{k_2,k_3,k_4,k_5\}-\max\{k_2,k_3,k_4,k_5\})}2^{-\frac{|k_6-k_7|}{2}}\prod_{i=1}^{7}||P_{k_i}\psi_i||_{S[k_i]}\end{aligned}$$

The remaining terms (32), (33) offer nothing new and are left out. Finally, the fifth and sixth therm in the above long list of error terms offer nothing new either. Note that one can place

$$\nabla^{-1}IP_k(R_\beta I\psi\nabla^{-1}(\psi\nabla^{-1}IQ_{\nu j}(\psi,\psi)))$$

into $L_t^\infty L_x^2$ or $L_t^\infty L_x^\infty$. This allows one to place the 'elliptic portion' of these terms into $L_t^2L_x^2$ by means of lemma 7.3, while the 'hyperbolic portion' can be put into $L_t^1L_x^2$. The reason why we can include the extra operator I in front of $P_k(...)$ has to do with the fact that if we apply an operator $(1-I)$ instead, we needn't apply dynamic separation to the first input of the quintilinear expression, as is easily verified. Putting together the estimates of sections 6, 7, 8 implies Proposition 4.1 in a standard way.

9. Appendix

9.1. **Proof of the energy inequality.**

Theorem 9.2. *Let ϕ, F be smooth functions on $[-T,T]\times\mathbf{R^2}$ such that $\Box\phi=F$. Then we have the inequality*

$$\begin{aligned}&||P_0\phi||_{S[0]([-T,T]\times\mathbf{R^2})}\\ &\lesssim \inf_{0<T_0\le T}(\min\{T_0,1\}^{-\frac{1}{M}}||P_0F||_{N[0]([-T,T]\times\mathbf{R^2})}+\sup_{t_0\in[-T_0,T_0]}||P_0\phi[t_0]||_{L_x^2})\end{aligned}$$

As usual we let $\phi[0]:=\{\partial_t\phi,\nabla_x\phi\}$.

Proof : Fix T_0. We may assume that F is a Schwartz function on $\mathbf{R}^{2+1}$. We subdivide

$$P_0F=P_0Q_{<0}F+P_0Q_{\geq 0}F$$

Then we observe that letting $\phi_2=\Box^{-1}P_0Q_{\geq 0}F$, where $\Box^{-1}$ is given by the multiplier $(-\tau^2+|\xi|^2)^{-1}$ on the (space-time) Fourier side, we first have straight from the definitions

$$||\nabla_{x,t}\Box^{-1}P_0Q_{\geq 0}F||_{\dot{X}_0^{-\frac{1}{2},0,2}} \lesssim ||P_0F||_{N[0]}.$$

Observe that when F is an atom of the first kind, this follows from Sobolev's inequality, and is trivial for atoms of the 2nd kind. If one uses in addition that

$$||Q_j\psi||_{L_t^\infty L_x^2} \lesssim 2^{\frac{j}{2}}||Q_j\psi||_{L_t^2L_x^2},\ ||P_0Q_{\geq 0}\psi||_{\dot{X}_0^{0,\frac{1}{2},\infty}} \lesssim ||P_0Q_{\geq 0}\psi||_{\dot{X}_0^{-\frac{1}{2},1,2}},$$

one deduces that the first three components of $||\phi_2||_{S[0]}$ are controlled. Next, one deduces

$$||\nabla_{x,t}\Box^{-1}(P_0Q_{\geq 0}F)||_{L_t^ML_x^2} \lesssim ||P_0F||_{N[0]}$$

This is immediate for the third kind of atoms, and follows via an interpolate of the inequality relating $||Q_j\psi||_{L_t^\infty L_x^2}$ and $||\psi||_{L_t^2L_x^2}$ for the 2nd and first class of atoms. In addition to giving control over the fourth component of $||\phi_0||_{S[0]}$, this means that there is a time slice $t=t_0\in[-T_0,T_0]$ where

$$||\nabla_{x,t}\Box^{-1}(P_0Q_{\geq 0}F)||_{L_x^2} \lesssim T_0^{-\frac{1}{M}}||P_0F||_{N[0]}$$

Now one constructs a solution of the equation

$$\Box\phi_1 = P_0Q_{<0}F,\ \phi_1[t_0]=0$$

via the truncated Duhamel's formula and obtains

$$P_0\phi = \phi_1+\phi_2+\phi_3$$

where ϕ_3 is a free wave with suitable initial conditions at $t=t_0$. As we have finished the proof for ϕ_2 (the fifth component of $||\phi_2||_{S[0]}$ is vacuous) and the estimate for ϕ_3 is standard (via [30] and the following remarks), we now focus on ϕ_1. We indicate here which modifications in the proof given in [30] are necessary in order to obtain our energy inequality. Relabel F to denote a Schwartz function coinciding with the $P_0Q_{<0}F$ on $[-T,T]\times\mathbf{R^2}$. The claim follows from the proof in [30] when F is either an $L_t^1\dot{H}_x^{-1}$ or a $\dot{X}_0^{-1,-\frac{1}{2},1}$-atom. Indeed, this is immediate for the first, 2nd and third constituent (since $P_kQ_{\geq k}\dot{X}_k^{-1,\frac{3}{2},\infty}\subset P_kQ_{\geq k}\dot{X}_k^{-\frac{1}{2},1,2}$). It follows for the fourth from an interpolate of the inequality $||Q_j\psi||_{L_t^\infty L_x^2}\lesssim 2^{\frac{j}{2}}||Q_j\psi||_{L_t^2L_x^2}$. Moreover, when F is an atom of the third class, we utilize the trivial estimate[37]

$$||F||_{\dot{X}_0^{-1,-\frac{1}{2},1}} \lesssim ||F||_{\dot{X}_0^{-\frac{1}{2},-1,2}},$$

[37]Keep the microsupport of F in mind.

in conjunction with the inequality (15). Thus we assume that F is a null-frame atom at frequency ~ 1 and angle 2^l, $l < -10$. By definition this means that we can decompose $F = \sum_{\kappa\in K_l} F_\kappa$, with F_κ microsupported in $\{\tau > 0, ||\tau| - |\xi|| \leq 2^{2l}, \frac{\xi}{|\xi|} \in \frac{\kappa}{2}\}$. Moreover, we have

$$(\sum_{\kappa\in K_l} ||F_\kappa||^2_{NFA[\kappa]})^{\frac{1}{2}} \leq 1$$

In this case, one computes ϕ_1 via Duhamel's formula, as in [30]. We only prove the result for the fifth component of $S[0]$, the other components following exactly as in [30]: indeed, in the latter work it is shown that one gets control over $||P_0 Q_{\geq 0}\phi_1||_{\dot{X}_0^{-1,\frac{3}{2},\infty}}$, and it is easy to see, using the 'Sobolev inequalities' of the first part of the proof, that this results in control over the third and fourth components of $||\phi_1||_{S[0]}$, while the first and 2nd require nothing new. Now let $\eta_T(t)$ be a smooth bump function supported on a dilate of $[-T, T]$ and identically equal to 1 on $[-T, T]$. Denote its restriction to $[0, \infty)$ by $\eta_T^+(t)$. It suffices to prove

$$\sup_{\lambda\geq l'} |\lambda|^{-1}(\sum_{\kappa\in K_{l'}} \sum_{R\in C_{0,\kappa,\lambda}} ||\tilde{P}_R Q^+_{<2l'} \int_{-\infty}^t \frac{\sin((t-s)\sqrt{-\triangle})}{\sqrt{-\triangle}} \eta_T^+(t-s)F(s)ds||^2_{S[0,\kappa]})^{\frac{1}{2}} \lesssim 1$$

because

$$\eta_T(t))S(t)(P_0\phi[0]) + \int_{-\infty}^t \frac{\sin((t-s)\sqrt{-\triangle})}{\sqrt{-\triangle}} \eta_T^+(t-s)P_0F(s)ds$$

is a Schwartz function and agrees with $P_0\phi$ on $[-T, T]$(the first summand is a truncated free wave, hence easy to control). Call the right-hand expression ψ, for simplicity's sake. Now we write $\psi = \sum_{\kappa\in K_l} \psi_\kappa$, where ψ_κ is defined in the same manner as ψ with F replaced by F_κ. We note the identity

$$\tilde{\psi_\kappa} = \frac{m_0(\xi)}{|\xi|}(\hat{\eta}_T^+(\tau - |\xi|) - \hat{\eta}_T^+(\tau - |\xi|))\tilde{F_\kappa}(\tau, \xi)$$

We let $\tilde{}$ refer to the space-time Fourier transform, whereas $\hat{}$ refers to Fourier transform with respect to either time- or space coordinates. In particular, $\tilde{\psi}$ is supported in the region $||\tau| - |\xi|| < 2^{2l}$. We need to show that for $\lambda \geq l'$

$$|\lambda|^{-1}(\sum_{\kappa'\in K_{l'}} \sum_{R\in C_{0,\kappa',\lambda}} ||\tilde{P}_R Q^+_{<2l'}\psi_\kappa||^2_{S[0,\kappa']})^{\frac{1}{2}} \lesssim (\sum_{\kappa\in K_l} ||F_\kappa||^2_{NFA[\kappa]})^{\frac{1}{2}}$$

We need to distinguish between the following cases:

(1): $l' > l + C$:

Using the crucial orthogonality property (16), we reduce the inequality in this case to the following: let $\omega \notin 2\kappa$:

$$|\lambda|^{-1}\Big(\sum_{R\in C_{0,\kappa,\lambda}}||\tilde{P}_R Q^{+}_{<2l+O(1)}\mathcal{F}^{-1}[\frac{m_0(\xi)}{|\xi|}(\hat{\eta}^{+}_{T}(\tau-|\xi|)+O(1))\tilde{F}_\kappa(\tau,\xi)]||^2_{S[0,\kappa]}\Big)^{\frac{1}{2}}$$
$$\lesssim \frac{1}{\mathrm{dist}(\omega,\kappa)}||F_\kappa||_{L^1_{t_\omega}L^2_{x_\omega}}$$

By the triangle inequality we may let $F(t_\omega, x_\omega) := \delta(t_\omega - t_0)f(x_\omega)$. Thus our new assertion implying the claim above is that

$$|\lambda|^{-1}\Big(\sum_{R\in C_{0,\kappa,\lambda}}||\tilde{P}_R Q^{+}_{<2l+O(1)}\mathcal{F}^{-1}[\frac{m_0(\xi)}{|\xi|}(\hat{\eta}^{+}_{T}(\tau-|\xi|)+O(1))\hat{f}(\xi_\omega)]||^2_{S[0,\kappa]}\Big)^{\frac{1}{2}}$$
$$\lesssim \frac{1}{\mathrm{dist}(\omega,\kappa)}||f||_{L^2}$$

However, the (flat)regions $R_{\in C_{0,\kappa,\lambda}} \cap C$, where C denotes the light cone, appear as thin curved strips in the ξ_ω reference frame, thanks to the curvature of the cone. Indeed, these strips have length $\sim 2^l$ and thickness $\sim 2^\lambda 2^{2l}$. In particular, thickening the flat strips $R_{\in C_{0,\kappa,\lambda}} \cap C$ by an amount $2^{2l+\lambda}$ will create only finite overlap of the corresponding regions in the ξ_ω reference frame. Letting $||f(x_\omega)||_{L^2_{x_\omega}} = 1$, the desired inequality will follow from the following:

$$\Big(\sum_{R\in C_{0,\kappa,\lambda}}||\tilde{P}_R Q^{+}_{2l+\lambda\leq .<2l+O(1)}\psi||^2_{S[k,\kappa]}\Big)^{\frac{1}{2}} \lesssim |\lambda|$$

$$\Big(\sum_{R\in C_{0,\kappa,\lambda}}||\tilde{P}_R Q^{+}_{<2l+\lambda}\mathcal{F}^{-1}[\frac{m_0(\xi)}{|\xi|}(\hat{\eta}^{+}_{T}(\tau-|\xi|)+O(1))\hat{f}(\xi_\omega)]||^2_{S[k,\kappa]}\Big)^{\frac{1}{2}} \lesssim 1$$

The first of these follows from the fact that $P_\kappa \dot{X}_0^{0,\frac{1}{2},1} \subset S[0,\kappa]$. As to the 2nd, observe that we can include a multiplier $\chi_R(\xi_\omega)$ which localizes to the support of $P_R Q_{<2l+\lambda}$ in the ξ_ω reference frame. Then we can invoke Plancherel with respect to x_ω to reduce to proving

$$||\tilde{P}_R Q^{+}_{<2l+\lambda}\mathcal{F}^{-1}[\frac{m_0(\xi)}{|\xi|}(\hat{\eta}^{+}_{T}(\tau-|\xi|)+O(1))\hat{f}(\xi_\omega)]||_{S[k,\kappa]} \lesssim ||f||_{L^2_{x_\omega}}$$

For this, get rid of the disposable multiplier $\tilde{P}_R$, which we replace by P_κ (recall $R \in C_{0,\kappa,\lambda}$). Then we reiterate application of the orthogonality property (16), reducing the claim to

$$(\sum_{\kappa'\in K_{l+\frac{\lambda}{2}-10},\,\kappa'\subset\kappa}||P_{0,\kappa'}Q^{+}_{<2l+\lambda}\mathcal{F}^{-1}[\frac{m_0(\xi)}{|\xi|}(\hat{\eta}^{+}_{T}(\tau-|\xi|)+O(1))\hat{f}(\xi_\omega)]||^2_{S[k,\kappa]})^{\frac{1}{2}}$$
$$\lesssim ||f||_{L^2_{x_\omega}}$$

This is proved in exactly this form in [30].

(2): $l-C<l'\leq l+C$: This is almost identical to the preceding case, hence left out.

(3): $2l<l'<l-C$: We need to show that if $\lambda\geq l'$, then

$$|\lambda|^{-1}(\sum_{\kappa'\subset\kappa,\,\kappa'\in K_{l'}}\sum_{R\in C_{0,\kappa',\lambda}}||\tilde{P}_R Q^{+}_{<2l'}\psi_k||^2_{S[0,\kappa']})^{\frac{1}{2}}\lesssim ||F_\kappa||_{NFA[\kappa]}$$

We may again assume that $F_\kappa=\delta(t_\omega-t_0)f(x_\omega)$ for some $\omega\notin 2\kappa$. As before, the supports of the operators $\tilde{P}_R Q^{+}_{<2l+l'}$ are finitely overlapping when projected onto the ξ_ω-plane. Moreover, we have $|(2l+\lambda)-2l'|\leq|l'|\leq|\lambda|$, whence the claim follows in the same way as before.

(4): $l'\leq 2l$: in this case, the operators $\tilde{P}_R Q^{+}_{<2l'}$ have finitely overlapping supports in the ξ_ω-reference system. One obtains the inequality without the $|\lambda|$-loss. ■

9.3. **Reducing various inputs of the quintilinear null-form to hyperbolic microsupport.** Recall from section 7 that the worst quintilinear term has the schematic form $\nabla_{x,t}[\psi\nabla^{-1}(R_\beta\psi\chi_\nu)]$. Our first task consists in reducing $R_\beta\psi$ to hyperbolic microsupport. First assume $\beta\neq 0$. For the subsequent discussion, let $I=\sum_{k\in\mathbf{Z}}P_kQ_{<k+100}$. Now we compute

$$||\nabla_{x,t}P_0Q_{>0}[P_{k_1}\psi\nabla^{-1}(R_\beta P_{k_2}(1-I)\psi P_{k_3}\chi_\nu)]||_{\dot{X}_0^{-\frac{1}{2},-1,2}}$$
$$\lesssim 2^{-\delta\max_{i=1,2,3}\{k_i\}}2^{-\delta|\min_i\{k_i\}-\max_i\{k_i\}|}\prod_{i=1,2}||P_{k_i}\psi||_{L^\infty_t L^2_x}[2^{\frac{k_3}{2}}||P_{k_3}\chi_\nu||_{L^2_tL^2_x}].$$

Then one uses lemma 7.3 proved. Similarly, we have

$$||\nabla_{x,t}P_0Q_{<0}[P_{k_1}\psi\nabla^{-1}(R_\beta P_{k_2}(1-I)\psi P_{k_3}\chi_\nu)]||_{L^1_tL^2_x}$$
$$\lesssim 2^{-\delta\max_{i=1,2,3}\{k_i\}}2^{-\delta|\min_i\{k_i\}-\max_i\{k_i\}|}\prod_{i=1,2}||P_{k_i}\psi||_{S[k_i]}[2^{\frac{k_3}{2}}||P_{k_3}\chi_\nu||_{L^2_tL^2_x}],$$

using the definition of $S[k]$. Next, assume that $\beta=0$. Then $\nu\neq 0$. Now we observe that if ψ solves the Wave Maps problem on $[-T,T]$, and if we substitute a Schwartz extension for ψ everywhere as in the preceding discussion, then

$$\begin{aligned}\nabla_{x,t}[\psi\nabla^{-1}(R_0\psi\chi_i)]|_{[-T,T]} &= \nabla_{x,t}[\psi\nabla^{-1}(R_0(1-I)\psi(\psi_i - R_i\sum_{k=1,2}R_k\psi_k))]|_{[-T,T]}\\ &\quad+\nabla_{x,t}[\psi\nabla^{-1}(R_0 I\psi\chi_i)]|_{[-T,T]}, \qquad (34)\end{aligned}$$

where χ_i is given by formula (11). The first summand on the right hand side in turn is morally equivalent to the schematic expression

$$\nabla_{x,t}[\psi\nabla^{-1}(R_0(1-I)\psi\psi)].$$

Freeze the output to frequency ~ 1. Now *either* the output is restricted to modulation > 1. Then we estimate

$$\begin{aligned}&||P_0Q_{>0}[P_{k_1}\psi\nabla^{-1}(R_0[1-I]P_{k_2}\psi P_{k_3}\psi)]||_{L_t^2L_x^2\cap L_t^ML_x^2}\\ &\qquad\lesssim 2^{-\delta\max_i\{k_i\}}2^{-\delta|\min_i\{k_i\}-\max_i\{k_i\}|}\prod_{i=1}^3||P_{k_i}\psi||_{S[k_i]},\end{aligned}$$

as before by placing $(1-I)P_{k_2}R_0\psi$ into $L_t^2L_x^2$ or $L_t^ML_x^2$ and considering simple frequency trichotomies in addition to Bernstein's inequality. *Or* the output is microlocalized to modulation ≤ 1. We now distinguish between the situations $k_3 \leq k_2 + O(1)$ and the opposite. In the former case, if $P_{k_3}\psi$ lives at modulation $< 2^{l-10}$, the product $R_0(1-I)P_{k_2}\psi P_{k_3}Q_{<l-10}\psi$ has elliptic microsupport. Then one argues as in the proof of theorem 6.10. If $P_{k_3}\psi$ has modulation $> 2^{l-10}$, one computes

$$\begin{aligned}&||P_0Q_{<0}[P_{k_1}\psi\nabla^{-1}P_a[R_0(1-I)P_{k_2}\psi P_{k_3}Q_{\geq l-10}\psi]]||_{L_t^1L_x^2}\\ &\lesssim 2^{-\delta|k_1|}2^{\delta(\min\{a,k_2,k_3\}-\max\{a,k_2,k_3\})}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]}\end{aligned}$$

for suitable $\delta > 0$ by placing $\nabla^{-1}P_a[R_0(1-I)P_{k_2}\psi P_{k_3}Q_{\geq l-10}\psi]$ into $L_t^1L_x^2 + L_t^1L_x^\infty$. In the latter case, i. e. $k_3 > k_2 + O(1)$, we rewrite the expression schematically as

$$\sum_{l>k_2+100}P_0Q_{<0}[P_{k_1}\psi\nabla^{-1}P_{k_3}\psi R_0(1-I)P_{k_2}Q_l\psi].$$

Then we distinguish between three possibilities:
(1): $k_2 \in [-100, 100]$. Then we have

$$\begin{aligned}&||P_0Q_{<0}[Q_{\geq l-10}(P_{k_1}\psi\nabla^{-1}P_{k_3}\psi)R_0(1-I)P_{k_2}Q_l\psi]||_{L_t^1L_x^2}\\ &\lesssim ||Q_{\geq l-10}(P_{k_1}\psi\nabla^{-1}P_{k_3}\psi)||_{L_t^2L_x^\infty}||R_0(1-I)P_{k_2}Q_l\psi||_{L_t^2L_x^2}\\ &\lesssim 2^{-\frac{l}{2}}2^{k_1-k_3}\prod_{i=1}^3||P_{k_i}\psi_i||_{S[k_i]},\end{aligned}$$

which can be summed over $l > O(1)$, as well as k_i. We have used lemma 6.4. On the other hand

$$\begin{aligned}
&||P_0 Q_{<0}[Q_{<l-10}(P_{k_1}\psi\nabla^{-1}P_{k_3}\psi)R_0(1-I)P_{k_2}Q_l\psi]||_{\dot{X}_0^{-1,-\frac{1}{2},1}} \\
&\lesssim ||Q_{<l-10}(P_{k_1}\psi\nabla^{-1}P_{k_3}\psi)||_{L_t^\infty L_x^\infty}||R_0(1-I)P_{k_2}Q_l\psi||_{L_t^2 L_x^2} \\
&\lesssim 2^{k_1-k_3}\prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}.
\end{aligned}$$

In this situation $l = O(1)$, and we can sum over k_i.
(2): $k_2 < -100$. Argue as in the immediately preceding but place $R_0(1-I)P_{k_2}\psi$ into $L_t^2 L_x^\infty$ to get an exponential gain in k_2.
(3): $k_2 > 100$. We need to gain exponentially in $-k_2$ in order to retrieve the frequency envelope ($k_1 = k_3 + O(1)$). For this, argue exactly as in case **(1)**, but place $Q_{\geq l-10}(P_{k_1}\psi\nabla^{-1}P_{k_3}\psi)$ into $L_t^2 L_x^2$. This finishes the reduction of $R_\beta\psi$ to hyperbolic microsupport.
For the case when it has 'hyperbolic microsupport', i. e. we apply I in front of it, we refer to (11) to replace χ_ν, as discussed in section 7. Then we enact dynamic separation within the resulting expression, arriving at a schematically written quintilinear expression of the following type:

$$\nabla_{x,t}[\psi\nabla^{-1}[R_\beta I\psi\nabla^{-1}[\psi\nabla^{-1}Q_{\nu j}(\psi,\psi)]]],$$

as well as error terms treated in section 8.. We now proceed analogously to theorem 6.10, and reduce $Q_{\nu j}(\psi,\psi)$ to 'hyperbolic microsupport': observe that if $l > k + 100$, then in an expression $\nabla^{-1}(P_{k_1}\psi P_k Q_l F)$, we have the following trichotomy:
(1): $k_1 \geq l - 80$.
(2): $P_{k_1}\psi$ at modulation $\geq 2^{l-100}$.
(3): $k_1 < l - 80$, $P_{k_1}\psi$ at modulation $< 2^{l-100}$.
Substituting $F = \nabla^{-1}Q_{\nu j}(\psi,\psi)$, we settle the situation corresponding to case **(2)** by means of the following 4 inequalities, which follow from lemma 6.2 and the definitions:

$$\begin{aligned}
&||\nabla^{-\epsilon}P_a[P_{k_1}Q_{\geq l-100}\psi P_k Q_l\nabla^{-1}Q_{\nu j}(P_{k_2}\psi, P_{k_3}\psi)]||_{L_t^{\frac{1}{1-\epsilon}}L_x^2} \\
&\lesssim 2^{\delta(\epsilon)[\min\{a,l,k,k_1\}-\max\{a,k,l,k_1\}]}2^{-\frac{|k_2-k_3|}{2}}\prod_{i=1,2,3}||P_{k_i}\psi||_{S[k_i]}
\end{aligned}$$

$$\begin{aligned}
&||\nabla^{-\epsilon}P_{a_1}[P_{a_2}\psi\nabla^{-(1-\epsilon)}P_{a_3}F]||_{L_t^1 L_x^{\frac{2}{1-2\epsilon}}} \\
&\lesssim 2^{\delta(\epsilon)[\min\{a_1,a_2,a_3\}-\max\{a_1,a_2,a_3\}]}||P_{a_2}\psi||_{S[a_2]}||P_{a_3}F||_{L_t^{\frac{1}{1-\epsilon}}L_x^2}
\end{aligned}$$

$$||P_0[P_{a_1}\psi\nabla^{-(1-\epsilon)}P_{a_2}F]||_{L_t^1 L_x^2} \lesssim 2^{-\delta|a_1|}||P_{a_1}\psi||_{S[a_1]}||P_{a_2}F||_{L_t^1 L_x^{\frac{2}{1-2\epsilon}}}$$

$$||\nabla^{-\frac{1}{2}}P_{a_1}[P_{a_2}\psi\nabla^{-1}P_{a_3}F]||_{L_t^2L_x^2}$$
$$\lesssim 2^{\delta[\min\{a_1,a_2,a_3\}-\max\{a_1,a_2,a_3\}]}||P_{a_2}\psi||_{S[a_2]}||P_{a_3}\nabla^{-\frac{1}{2}}F||_{L_t^2L_x^2}$$

From the first three of these, we deduce

$$\begin{aligned}&||\nabla_{x,t}P_0Q_{<0}[P_{a_1}\psi\nabla^{-1}P_{a_2}[R_\beta IP_{a_3}\psi\\ &\qquad\nabla^{-1}P_{a_4}[P_{a_5}Q_{\geq l-100}\psi\nabla^{-1}P_{a_6}Q_lQ_{\nu j}(P_{a_7}\psi,P_{a_8}\psi)]]]]||_{L_1^1L_x^2}\\ &\lesssim 2^{-\delta|a_1|}2^{\delta[\min\{l,a_2,a_3,a_4,a_5,a_6\}-\max\{l,a_2,a_3,a_4,a_5,a_6\}]}2^{-|\frac{a_7-a_8|}{2}}\prod_{i=1,3,5,7,8}||P_{a_i}\psi||_{S[a_i]}.\end{aligned}$$

One can now sum over all frequency parameters and l. When the output is reduced to large modulation, we use theorem 6.10 and the fourth of the above inequalities. Next, assume we are in situations **(1)** or **(3)**. Reiterating the same trichotomy, and observing that in case **(12)** or **(32)** we can argue as before, we are left with 4 situations, leading to the following schematically written terms:
(11): $\nabla_{x,t}[\psi\nabla^{-1}P_{>l-80}(\psi\nabla^{-1}P_{>l-80}\psi)\nabla^{-1}P_kQ_lQ_{\nu j}(\psi,\psi)]$.
(13): $\nabla_{x,t}[\psi\nabla^{-1}P_{<l-80}Q_{l+O(1)}[\psi\nabla^{-1}P_{>l-80}\psi\nabla^{-1}P_kQ_lQ_{\nu j}(\psi,\psi)]]$.
(31): $\nabla_{x,t}[\psi\nabla^{-1}P_{>l-80}\psi\nabla^{-1}P_{<l-80}Q_{l+O(1)}[\psi\nabla^{-1}P_kQ_lQ_{\nu j}(\psi,\psi)]]$.
(33): $\nabla_{x,t}[\psi\nabla^{-1}P_{<l-80}Q_{l+O(1)}[\psi\nabla^{-1}(\psi\nabla^{-1}P_kQ_lQ_{\nu j}(\psi,\psi))]]$.
Consider the first of these terms: we commence with the case $l>90$. We estimate it by invoking lemma 6.2the inequalities

$$||P_a[\psi\nabla^{-1}P_{>l-80}(\psi\nabla^{-1}P_{>l-80}\psi)]||_{L_t^2L_x^{2+}}\lesssim 2^{\mu a}\sum_{b>l-80}2^{-\nu b}||P_b\psi||_{S[b]}$$

$$||P_a[\psi\nabla^{-1}P_{>l-80}(\psi\nabla^{-1}P_{>l-90}\psi)]||_{L_t^\infty L_x^2}\lesssim 2^{\mu a}\sum_{b>l-80}2^{-\nu b}||P_b\psi||_{S[b]},$$

for suitable μ, $\nu>0$, which follow as in section 7 by using improved Strichartz type norms, as well as

$$||P_0Q_{\geq l}[\psi\nabla^{-1}P_{>l-80}(\psi\nabla^{-1}P_{>l-80}\psi)]||_{\dot{X}_0^{0,1-\epsilon,1}}\lesssim\sum_{b>l-80}2^{-\mu b}||P_b\psi||_{S[b]},$$

which is gotten by reiterating the proof of lemma 6.4. One can then argue as in the proof of theorem 6.10. Now assume $l<90$, whence $k<-10$. We observe the following inequality[38], provided $a_3>l-80$,

$$\begin{aligned}&||P_0Q_{<l-10}[P_{a_1}\psi\nabla^{-1}P_{>l-80}(P_{a_2}\psi\nabla^{-1}P_{a_3}\psi)]||_{L_t^\infty L_x^2}\\ &\lesssim 2^{-\delta\min\{|a_1|,|a_3|\}}2^{\delta(\min\{\min\{a_1,a_2,0\},a_3\}-\max\{\min\{a_1,a_2,0\},a_3\})}\prod_{i=}^{3}||P_{a_i}\psi_i||_{S[a_i]}.\end{aligned}$$

This is immediate provided $a_2<a_3+O(1)$. In the case $a_2>>a_3$, we may assume $a_3<-10$, as is easily seen, and we replace the expression above by

[38]In the following, δ denotes a small positive number, depending on the context.

$P_0Q_{<l-10}[P_{a_1}\psi\nabla^{-1}P_{a_2}\psi\nabla^{-1}P_{a_3}\psi)]$. Then we estimate

$$\begin{aligned}
&||P_0Q_{<l-10}[P_{a_1}\psi Q_{<l-10}(P_{a_2}\nabla^{-1}\psi\nabla^{-1}P_{a_3}\psi)]||_{L_t^\infty L_x^2}\\
&\lesssim 2^{\min\{\frac{l-\min\{a_1,a_2,0\}}{4+},0\}}2^{\min\{a_1-a_2,0\}}||P_{a_1}||_{S[a_1]}||Q_{<l-10}(P_{a_2}\psi\nabla^{-1}P_{a_3}\psi)||_{\dot{X}_{a_2}^{0,\frac{1}{2},1}}\\
&\lesssim 2^{\min\{\frac{l-\min\{a_1,a_2,0\}}{4+},0\}}2^{\frac{l-a_3}{4+}}2^{\min\{a_1-a_2,0\}}\prod_{i=1,2,3}||P_{a_i}\psi||_{S[a_i]}.
\end{aligned}$$

Since $l<\min\{a_2,a_3\}+O(1)$, the inequality follows in this case. Similarly assuming $a_1\geq 0$

$$\begin{aligned}
&||P_0Q_{<l-10}[P_{a_1}\psi Q_{\geq l-10}(P_{a_2}\nabla^{-1}\psi\nabla^{-1}P_{a_3}\psi)]||_{L_t^\infty L_x^2}\\
&\lesssim 2^{\frac{l}{2}}||P_{a_1}\psi||_{L_t^\infty L_x^2}||Q_{\geq l-10}(P_{a_2}\nabla^{-1}\psi\nabla^{-1}P_{a_3}\psi)||_{L_t^2L_x^\infty}\\
&\lesssim 2^{\frac{l}{2}}\sum_{b\geq l-10}2^{-\frac{b}{2}}2^{\min\{\frac{b-a_2}{4},0\}}2^{\min\{\frac{b-a_3}{4+},0\}}\prod_{i=1}^3||P_{a_i}\psi||_{S[a_i]}
\end{aligned}$$

The case $a_1<0$ is treated analogously, and the claim follows again. Now we can compute under our current assumptions on k, l

$$\begin{aligned}
&||P_0\nabla_{x,t}[Q_{<l-10}[P_{a_1}\psi\nabla^{-1}P_{>l-80}(P_{a_2}\psi\nabla^{-1}P_{>l-80}P_{a_3}\psi)]\\
&\qquad\qquad\nabla^{-1}P_kQ_lQ_{\nu j}(P_{a_4}\psi,P_{a_5}\psi)]||_{\dot{X}_0^{0,-\frac{1}{2},1}}\\
&\lesssim 2^{-\delta\min\{|a_1|,\,|a_3|\}}2^{\delta(\min\{\min\{a_1,a_2,0\},a_3\}-\max\{\min\{a_1,a_2,0\},a_3\})}\\
&\qquad\qquad 2^{\frac{k-l}{2}}2^{\frac{a_4-a_5}{2}}\prod_{i=1}^5||P_{a_i}\psi||_{S[a_i]}
\end{aligned}$$

One verifies that summation over all frequency parameters is possible and reproduces the frequency envelope, except possibly summation over l. However, a quick inspection of the proof of lemma 6.2 reveals that either $a_4=a_5+O(1)=l+O(1)$, or else at least one of the inputs of $Q_{\nu j}(\psi,\psi)$ lives at modulation at least $\sim 2^l$, in which case one gains exponentially in both $k-\max\{a_4,a_5\}$ as well as $k-l$; both situations suffice to sum over l. By exact analogy, we deduce for $a_3>l-80$ the estimate

$$\begin{aligned}
&||P_0Q_{[l-10,l+10]}[P_{a_1}\psi\nabla^{-1}P_{>l-80}(P_{a_2}\psi\nabla^{-1}P_{a_3}\psi)]||_{\dot{X}_0^{0,\frac{1}{2},1}}\\
&\lesssim 2^{-\delta\min\{|a_1|,\,|a_3|\}}2^{\delta(\min\{\min\{a_1,a_2,0\},a_3\}-\max\{\min\{a_1,a_2,0\},a_3\})}\prod_{i=}^3||P_{a_i}\psi_i||_{S[a_i]}.
\end{aligned}$$

One deduces from this that

$$||P_0\nabla_{x,t}[Q_{[l-10,l+10]}[P_{a_1}\psi\nabla^{-1}P_{>l-80}(P_{a_2}\psi\nabla^{-1}P_{>l-80}P_{a_3}\psi)]$$
$$\nabla^{-1}P_kQ_lQ_{\nu j}(P_{a_4}\psi,P_{a_5}\psi)]||_{L_t^1L_x^2}$$
$$\lesssim 2^{-\delta\min\{|a_1|,|a_3|\}}2^{\delta(\min\{\min\{a_1,a_2,0\},a_3\}-\max\{\min\{a_1,a_2,0\},a_3\})}$$
$$2^{\frac{k-l}{2}}2^{\frac{a_4-a_5}{2}}\prod_{i=1}^{5}||P_{a_i}\psi||_{S[a_i]}$$

and proceeds as above. The case $Q_{>l+10}[P_{a_1}\psi\nabla^{-1}P_{>l-80}(P_{a_2}\psi\nabla^{-1}P_{>l-80}P_{a_3}\psi)]$ is similar. This concludes the treatment for case **(11)**. For the term **(13)**, apply theorem 6.10 to place

$$\nabla^{-1}P_{>l-80}\psi\nabla^{-1}P_kQ_lQ_{\nu j}(\psi,\psi)$$

into $L_t^2L_x^2$:

$$2^{\frac{a_1}{2}}||P_{a_1}[\nabla^{-1}P_{>l-80}P_{a_2}\psi\nabla^{-1}P_kQ_lQ_{\nu j}(P_{a_3}\psi,P_{a_4}\psi)]||_{L_t^2L_x^2}$$
$$\lesssim 2^{\delta(\min\{a_1,k,a_2\}-\max\{a_1,k,a_2\})}2^{-\frac{|a_3-a_4|}{2}}\prod_{i=2,3,4}||P_{a_i}\psi||_{S[a_i]}.$$

Indeed, the same inequality obtains if we square-sum over l. Combine this with the inequality

$$2^{\frac{a_1}{2}}||P_{a_1}\nabla^{-1}[P_{a_2}\psi P_{a_3}F]||_{L_t^2L_x^2}$$
$$\lesssim 2^{\delta(\min\{a_1,a_2,a_3\}-\max\{a_1,a_2,a_3\})}||P_{a_2}\psi||_{S[a_2]}[2^{\frac{a_3}{2}}||P_{a_3}F||_{L_t^2L_x^2}]$$

We can now compute as in the proof of theorem 6.10

$$||\sum_{l>100}\nabla_{x,t}P_0[P_{a_1}Q_{<l-10}\psi\nabla^{-1}P_{<l-80}Q_{l+O(1)}P_{a_2}[P_{a_3}\psi$$
$$\nabla^{-1}P_{>l-80}P_{a_4}\psi\nabla^{-1}P_{a_4}Q_lQ_{\nu j}(P_{a_5}\psi,P_{a_6}\psi)]]||_{\dot{X}_0^{-\frac{1}{2},-1,2}}$$
$$\lesssim 2^{\delta(\min_{i=1}^4\{a_i\}-\max_{i=1}^4\{a_i\})}2^{-\frac{|a_5-a_6|}{2}}\prod_{i=1}^{6}||P_{a_i}\psi||_{S[a_i]}$$

The case when the first input $P_{a_1}\psi$ is elliptic, as well as the situation $l\leq 100$, are treated in analogy to earlier computations. For **(31)**, we argue as for **(11)** to reduce l to size $< O(1)$. We need to estimate two expressions, the first of which is

$$||\nabla_{x,t}[P_{O(1)}Q_{<l-C}[P_{a_1}\psi\nabla^{-1}P_{>l-80}P_{a_2}\psi]$$
$$\nabla^{-1}P_{<l-80}Q_{l+O(1)}P_{a_3}[P_{a_4}\psi\nabla^{-1}P_{a_5}Q_lQ_{\nu j}(P_{a_6}\psi,P_{a_7}\psi)]]||_{\dot{X}_0^{0,-\frac{1}{2},1}}$$
$$\lesssim 2^{\delta(l-\min\{a_2,0\})}2^{\min\{a_1-a_2,0\}}$$
$$2^{\delta(\min\{a_3,a_4,a_5\}-\max\{a_3,a_4,a_5\})}2^{\frac{a_3-l}{2}}2^{-\frac{|a_6-a_7|}{2}}\prod||P_{a_i}\psi||_{S[a_i]}.$$

It is easy to check that one can sum over all parameters, and recovering the frequency envelope in the end. The case when $P_{a_1}\psi\nabla^{-1}P_{>l-80}P_{a_2}\psi$ is at modulation $\geq 2^{l-C}$ is treated analogously. Finally, the case **(33)** is similar to case **(13)**.

9.4. **Finishing the treatment of the quintilinear terms.** Referring to the original expression for $N(\psi_1,\dots,\psi_5)$, frequency localizing the ψ_i to frequency $\sim 2^{k_i}$, applying the further reductions on modulations and frequencies as discussed in the first paragraph of the proof of Proposition 7.2 and finally assuming that $k_2 >> k_3$, we distinguish between the following cases:
(1): *High-high interaction between* $P_{k_2}\psi_2$ *and* $(\sum_i \triangle^{-1}\partial_i\dots)$. One arrives at the following three schematic types of expressions:

$$\partial^\nu P_{k_1}\psi_1\nabla^{-1}P_k(\nabla^{-1}P_{k_2}I\psi_2R_\nu P_{k_3}\psi_3\nabla^{-1}IQ_{ik}(P_{k_4}\psi_4,P_{k_5}\psi_5))$$

$$\partial^\nu P_{k_1}\psi_1\nabla^{-1}P_k(\nabla^{-1}P_{k_2}I\psi_2P_{k_3}\psi_3\nabla^{-1}IQ_{\nu j}(P_{k_4}\psi_4,P_{k_5}\psi_5))$$

$$\partial^\nu P_{k_1}\psi_1\nabla^{-1}P_k(R_\nu P_{k_2}I\nabla^{-1}\psi_2P_{k_3}\psi_3\nabla^{-1}IQ_{ik}(P_{k_4}\psi_4,P_{k_5}\psi_5)).$$

These are treated like the fourth, first and (a minor variation of) the fourth term of A_3 in the proof of Proposition 7.2, respectively.
(2): *High-Low interaction between* $P_{k_2}\psi_2$ *and* $(\sum_i \triangle^{-1}\partial_i\dots)$. Observe that we may assume $k_2 < -10$. One arrives at the following three types of expressions:

$$\partial^\nu P_{k_1}\psi_1\nabla^{-1}P_{k_2}I\psi_2\nabla^{-1}P_{<k_2}[R_\nu P_{k_3}\psi_3\nabla^{-1}IQ_{ik}(P_{k_4}\psi_4,P_{k_5}\psi_5)]$$

$$\partial^\nu P_{k_1}\psi_1\nabla^{-1}P_{k_2}I\psi_2\nabla^{-1}P_{<k_2}[P_{k_3}\psi_3\nabla^{-1}IQ_{\nu j}(P_{k_4}\psi_4,P_{k_5}\psi_5)]$$

$$\partial^\nu P_{k_1}\psi_1\nabla^{-1}P_{k_2}R_\nu I\psi_2\nabla^{-1}P_{<k_2}[P_{k_3}\psi_3\nabla^{-1}IQ_{ik}(P_{k_4}\psi_4,P_{k_5}\psi_5)].$$

The first term is treated in close analogy to the last two terms of A_3. The 2nd is similar to the first term of A_3. $P_{k_3}\psi_3$ has a role analogous to a bilinear expression in the estimation of the latter term. This is somewhat worse, however, inspection of the estimates in the proof of Proposition 7.2 reveals that we are still better off than in the trilinear estimates, since we substitute a bilinear expression (namely $\nabla^{-1}P_{k_4}\psi_4P_{k_5}\psi_5$) for one input in the corresponding trilinear estimate. The last of the three terms above is similar to the third term of A_3.
(3): *Low-High interactions between* $P_{k_2}\psi_2$ *and* $(\sum_i \triangle^{-1}\partial_i\dots)$. This is treated similarly.

References

[1] P. D'Ancona, V.Georgiev *On the continuity of the solution operator of the wave maps system*, preprint

[2] P. Bizon, Comm.Math.Phys.215(2000), 45

[3] D.Christodoulou, A. Tahvildar-Zadeh *On the regularity of spherically symmetric wave maps*, C.P.A.M., 46(1993), 1041-1091

[4] M. Guenther, *Isometric embeddings of Riemannian manifolds*, Proceedings Interntl. Congress of Mathematicians, Vol. 1-2 (Kyoto 1990),p.1137-1143, Tokyo, 1991, Math. Soc. Japan.
[5] C.-H. Gu, *On the Cauchy problem for harmonic maps defined on two-dimensional Minkowski space*, Comm. Pure Appl. Math. 33: 727-737, 1980.
[6] F.Helein, *Regularite des applications faiblement harmoniques entre une surface et une varietee Riemanienne*, C.R.Acad.Sci.Paris Ser.1 Math 312(1991), 591-596
[7] S.Klainerman, *UCLA lectures on nonlin. wave eqns.*, preprint (2001)
[8] S.Klainerman, D.Foschi, *Bilinear Space-Time Estimates for Homogeneous Wave Equations*, Ann. Scient. Ec. Norm. Sup., 4e serie, t.33(2000), 211-274
[9] S.Klainerman, M.Machedon, *Smoothing estimates for null forms and applications*, Duke Math.J., 81(1995), 99-133
[10] S.Klainerman, M.Machedon, *On the algebraic properties of the $H^{\frac{n}{2},\frac{1}{2}}$ spaces*, I.M.R.N. 15(1998), 765-774
[11] S.Klainerman, M.Machedon, *On the regularity properties of a model problem related to wave maps*, Duke Math.J., 87(1997), 553-589
[12] S.Klainerman, I.Rodnianski, *On the global regularity of wave maps in the critical Sobolev norm*, I.M.R.N. 13(2001), 655-677
[13] S.Klainerman, S.Selberg, *Remark on the optimal regularity for equations of wave maps type*, C.P.D.E., 22(1997), 901-918
[14] S.Klainerman, S.Selberg, *The spaces $H^{s,\theta}$ and applications to nonlinear wave equations.*, preprint
[15] S.Klainerman, S.Selberg, *Bilinear estimates and applications to nonlinear wave equations*, preprint
[16] S.Klainerman, D.Tataru, *On the optimal regularity for the Yang-Mills equations in $\mathbf{R}^{\mathbf{4+1}}$*, Journal of the American Math. Soc., 12(1999), 93-116
[17] J.Krieger, *Global Regularity of Wave Maps in 2 and 3 spatial dimensions*, Ph. D. Thesis, Princeton University (2003)
[18] J. Krieger, *Global regularity of Wave Maps from $\mathbf{R}^{3+1}$ to surfaces*, CMP 238/1-2 (2003), 333-366
[19] J.Krieger, *Null-Form estimates and nonlinear waves*, to appear in Advances in Difftl. Eqns.
[20] A.Nahmod, A.Stefanov, K.Uhlenbeck, *On the well-posedness of the wave maps problem in high dimensions*, preprint(2001)
[21] S.Selberg,*Multilinear space-time estimates and applications to local existence theory for nonlinear wave equations*, Ph.D. thesis, Princeton University, 1999
[22] J.Shatah, A. Tahvildar-Zadeh, *On the Cauchy Problem for Equivariant Wave Maps*, Comm. Pure Appl. Math. 47(1994), 719-754
[23] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993
[24] M.Struwe, J.Shatah, *The Cauchy problem for wave maps*, I.M.R.N.11(2002), 555-571
[25] M.Struwe, J.Shatah, *Geometric Wave Equations*, AMS Courant Lecture Notes 2
[26] M.Struwe, *Equivariant Wave Maps in 2 space dimensions*, preprint
[27] M.Struwe, *Radially Symmetric Wave Maps from 1+2 dimensional Minkowski space to the sphere*, Math.Z.242(2002)
[28] T.Tao, *Ill-posedness for one-dimensional Wave Maps at the critical regularity*, Am. Journal of Math.122 No.3(200), 451-463
[29] T.Tao, *Global regularity of wave maps I*, I.M.R.N. 6(2001), 299-328
[30] T.Tao, *Global regularity of wave maps II*, Comm.Math.Phys.224(2001), 443-544
[31] T.Tao, *Counterexamples to the n=3 endpoint Strichartz estimate for the wave equation*, preprint
[32] D.Tataru, *Local and global results for wave maps I*, Comm. PDE 23(1998), 1781-1793
[33] D.Tataru,*On global existence and scattering for the wave maps equation*, Amer. Journal. Math.123(2001), no.1, 37-77
[34] D. Tataru, *Rough solutions for the Wave Maps equation*, preprint